

ON THE EXISTENCE OF WEAK SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS AND SYSTEMS WITH HARDY POTENTIALS

KONSTANTINOS T. GIKAS AND PHUOC-TAI NGUYEN

ABSTRACT. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded smooth domain and $\delta(x) = \text{dist}(x, \partial\Omega)$. In this paper, we provide various necessary and sufficient conditions for the existence of weak solutions to

$$-\Delta u - \frac{\mu}{\delta^2} u = \delta^\gamma u^p + \tau \quad \text{in } \Omega, \quad u = \nu \quad \text{on } \partial\Omega$$

where $\mu, \gamma \in \mathbb{R}$, $p > 0$, τ and ν are measures on Ω and $\partial\Omega$ respectively. We then establish existence results for the system

$$\begin{cases} -\Delta u - \frac{\mu}{\delta^2} u = \epsilon \delta^\gamma v^p + \tau & \text{in } \Omega, \\ -\Delta v - \frac{\mu}{\delta^2} v = \epsilon \delta^{\tilde{\gamma}} u^{\tilde{p}} + \tilde{\tau} & \text{in } \Omega, \\ u = \nu, \quad v = \tilde{\nu} & \text{on } \partial\Omega \end{cases}$$

where $\epsilon = \pm 1$, $\gamma, \tilde{\gamma} \in \mathbb{R}$, $p > 0$, $\tilde{p} > 0$, τ and $\tilde{\tau}$ are measures on Ω , ν and $\tilde{\nu}$ are measures on $\partial\Omega$. We also deal with elliptic systems where the nonlinearities are more general.

Key words: Hardy potential, semilinear equations, elliptic systems, normalized boundary trace.

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Email address: kgkikas@dim.uchile.cl.

Email address: nguyenphuoc tai.hcmup@gmail.com.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded smooth domain (say $\partial\Omega \in C^2$), $\delta(x)$ be the distance from $x \in \Omega$ to $\partial\Omega$ and $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. In the present paper we study semilinear problems with *Hardy potential* of the form

$$(1.1) \quad \begin{cases} -\Delta u - \frac{\mu}{\delta^2} u = h(x, u) + \tau & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega \end{cases}$$

where $\mu \in \mathbb{R}$, τ and ν are Radon measures on Ω and $\partial\Omega$ respectively.

The case without Hardy potential and with power absorption nonlinearity, i.e. $\mu = 0$, $\tau = 0$, $h(x, u) = -|u|^{p-1}u$, $p > 1$, is well understood in the literature, starting with a work by Gmira and Véron [19]. It was proved that there is a critical exponent $p^* := \frac{N+1}{N-1}$ in the sense that if $p \in (1, p^*)$ then there is a unique weak solution for every finite measure ν on Ω , while if $p \in [p^*, \infty)$ there exists no solution with isolated boundary singularity. Marcus and Véron [26] studied this problem by introducing a notion of *boundary trace* and provided a complete description of isolated singularities in the *subcritical case*, i.e. $1 < p < p^*$. The *supercritical case*, i.e. $p \geq p^*$, requires a deeper analysis and was studied by several authors. In particular, the problem was studied by Dynkin [15], Mselati [24] in the case $p^* \leq p \leq 2$ using probabilistic techniques, and by Marcus and Véron [27, 28] for $p \geq p^*$ due to analytic methods.

The solvability of the semilinear problem without Hardy potential and with *power source term*, namely $\mu = 0$, $\tau = 0$, $h(x, u) = u^p$, $p > 1$, was first studied by Bidaud-Véron and Vivier [8] in connection with the Green operator \mathbb{G} and the Poisson operator \mathbb{P} associated to $(-\Delta)$ in Ω . They proved that, in the subcritical case $1 < p < p^*$, the problem admits a solution if and only if the total mass of ν is sufficiently small. Afterwards, Bidaud-Véron and Yarur [10] reconsidered this type of problem in a more general setting and provided a necessary and sufficient condition for the existence of solutions. In the same spirit, (1.1) with $\mu = 0$, $h(x, u) = u^p$, $p > 0$, was also considered Bidaud-Véron and Yarur in [10]. Recently, Bidaud-Véron et al. [9] provided new criteria for the existence of solutions with $p > 1$ in terms of Besov capacity $\text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}$ associated to the Besov space $B_{\frac{2}{p}, p'}^{\frac{2}{p}, p'}(\mathbb{R}^{N-1})$. More precisely, it was pointed out in [9] that the solvability can be expressed by the inequality $\nu(E) \leq C \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(E)$ for every Borel subset $E \subset \partial\Omega$. Problem (1.1) with $\mu = 0$, $\tau = 0$ and $h(x, u)$ depending only on u and satisfying a so-called *subcriticality condition* was investigated by Chen, Felmer and Véron [13] due to Schauder fixed point theorem in Marcinkiewicz setting.

Recently, semilinear equations with Hardy potential depending on the distance function have been attracted much attention.

Put

$$L_\mu := \Delta + \frac{\mu}{\delta^2}.$$

Let $\phi \geq 0$ in Ω and $p \geq 1$, we denote by $L^p(\Omega; \phi)$ the space of all function v on Ω satisfying $\int_\Omega |v| \phi dx < \infty$. We denote by $\mathfrak{M}(\Omega; \phi)$ the space of Radon measures τ on Ω satisfying $\int_\Omega \phi d|\tau| < \infty$ and by $\mathfrak{M}^+(\Omega; \phi)$ the nonnegative cone of $\mathfrak{M}(\Omega; \phi)$. When $\phi \equiv 1$, we use the notations $\mathfrak{M}(\Omega)$ and $\mathfrak{M}^+(\Omega)$. We also denote by $\mathfrak{M}(\partial\Omega)$ the space of finite measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the nonnegative cone of $\mathfrak{M}(\partial\Omega)$.

Let G_μ and K_μ be the Green kernel and Martin kernel of $-L_\mu$ in Ω , \mathbb{G}_μ and \mathbb{K}_μ be the corresponding Green operator and Martin operator. Let C_H be Hardy constant, namely

$$(1.2) \quad C_H = \inf_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega (v/\delta)^2 dx}$$

then it is well known that $0 < C_H \leq \frac{1}{4}$ and if Ω is convex then $C_H = \frac{1}{4}$ (see for example [21]). Moreover the infimum is achieved if and only if $C_H < \frac{1}{4}$. When $\mu < C_H$ or $-\Delta\delta \geq 0$ in Ω in the sense of distribution, the first eigenvalue $\lambda_{\mu,1}$ of L_μ in Ω exists. For $\mu \in (0, \frac{1}{4}]$, denote by α_+ and α_- two fundamental exponents

$$(1.3) \quad \alpha_\pm := \frac{1}{2}(1 \pm \sqrt{1 - 4\mu})$$

then $0 < \alpha_- < \frac{1}{2} < \alpha_+ < 1$ and $\alpha_+ + \alpha_- = 1$. The eigenfunction $\varphi_{\mu,1}$ associated to $\lambda_{\mu,1}$ with the normalization $\int_\Omega (\varphi_{\mu,1}/\delta)^2 dx = 1$ satisfies (see [12], [16]), for some constant $c > 0$,

$$(1.4) \quad c^{-1}\delta^{\alpha_+} \leq \varphi_{\mu,1} \leq c\delta^{\alpha_+} \quad \text{in } \Omega.$$

In relation to Hardy constant, Bandle et al. in [6] classified large solutions of the linear equation

$$(1.5) \quad -L_\mu u = 0 \quad \text{in } \Omega,$$

and the associated nonlinear equation with power absorption

$$(1.6) \quad -L_\mu u + u^p = 0 \quad \text{in } \Omega.$$

In [23], Marcus and P.-T. Nguyen studied boundary value problem for (1.5) and (1.6) with $\mu \in (0, C_H)$ in measures framework by introducing a notion of *normalized boundary trace* which is defined as follows

Definition 1.1. *A function u possesses a normalized boundary trace if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that*

$$(1.7) \quad \lim_{\beta \rightarrow 0} \beta^{-\alpha_-} \int_{\{x \in \Omega : \delta(x) = \beta\}} |u - \mathbb{K}_\mu[\nu]| dS = 0.$$

The normalized boundary trace is denoted by $\text{tr}^(u)$.*

The restriction $\mu \in (0, C_H)$ in [23] is due to the fact that in this case L_μ is *weakly coercive* in $H_0^1(\Omega)$ and consequently by a result of Ancona [3, Remark p. 523] there is a $(1-1)$ correspondence between $\mathfrak{M}^+(\partial\Omega)$ and the class of positive L_μ harmonic functions, namely any positive L_μ harmonic function u can be written in a unique way under the form $u = \mathbb{K}_\mu[\nu]$ for some $\nu \in \mathfrak{M}^+(\partial\Omega)$.

The notion of normalized boundary trace was proved [23] as an appropriate generalization of the classical boundary trace to the setting of Hardy potentials, giving a characterization of *moderate solutions* of (1.6). In addition, it was showed in [23] that there exists a critical exponent $p_\mu^* := \frac{N+\alpha_+}{N+\alpha_+-2}$ such that if $p \in (0, p_\mu^*)$ then there exists a unique solution of (1.6) with $\text{tr}^*(u) = \nu$ for every finite measure ν on $\partial\Omega$, while if $p > p_\mu^*$ there is no solution of (1.6) with isolated boundary singularity. Marcus and Moroz [22] then *extended the notion of normalized boundary trace to the case $\mu < \frac{1}{4}$* and employed it to investigate (1.6). They proved that the existence depends on two critical exponents and discussed the question of uniqueness. Separable solutions of (1.6) with $\mu < \frac{1}{4}$ with a strong boundary singularity was studied in [5]. When $\mu = \frac{1}{4}$, L_μ is no longer weakly coercive and hence Ancona's result cannot be applied. However, Gkikas and Véron [18] observed that *if the first eigenvalue of $-L_{\frac{1}{4}}$ is positive* then the kernel $K_{\frac{1}{4}}(x, y)$ with pole at $y \in \partial\Omega$ is unique up to a multiplication and any positive $L_{\frac{1}{4}}$ harmonic function u admits such a representation. Based on that observation, they considered the boundary value problem with measures for (1.6), fully classifying isolated singularities in the subcritical case $p \in (1, p_\mu^*)$ and providing removability result in the supercritical case $p \geq p_\mu^*$.

In parallel, semilinear equations with Hardy potential and source term

$$(1.8) \quad -L_\mu u = u^p \quad \text{in } \Omega$$

were treated by Bidaut-Véron et al. [9] and by P.-T. Nguyen [31] and a fairly complete description of the profile of solutions to (1.8) was obtained in subcritical case $p < p_\mu^*$ (see [31]) and in supercritical case $p \geq p_\mu^*$ (see [9]).

Motivated by the above works, we aim to obtain existence results for (1.1). Throughout this paper, we use the notation $(h \circ u)(x) := h(x, u(x))$. Before stating main results, it is necessary to give the definition of weak solutions to (1.1). Put

$$(1.9) \quad \mathbf{X}(\Omega) = \{ \phi \in L^2(\Omega) \text{ s.t. } \nabla(\delta^{-\alpha+} \phi) \in L^2(\Omega, \delta^{\alpha+}) \text{ and } \delta^{-\alpha+} L_\mu \phi \in L^\infty(\Omega) \}.$$

Definition 1.2. Assume $\mu \in (0, \frac{1}{4}]$. Let $\tau \in \mathfrak{M}(\Omega; \delta^{\alpha+})$ and $\nu \in \mathfrak{M}(\partial\Omega)$. A function u is called a weak solution of (1.1) if $u \in L^1(\Omega; \delta^{\alpha+})$, $h \circ u \in L^1(\Omega; \delta^{\alpha+})$ and for every $\phi \in \mathbf{X}(\Omega)$,

$$(1.10) \quad - \int_\Omega u L_\mu \phi \, dx = \int_\Omega (h \circ u) \phi \, dx + \int_\Omega \phi d\tau - \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \phi \, dx.$$

for every $\phi \in \mathbf{X}(\Omega)$

Remark. (i) It is showed in [18] that a function u is a weak solution of (1.1) if and only if it can be written in the following form

$$(1.11) \quad u = \mathbb{G}_\mu[h \circ u] + \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu].$$

(ii) It is worth mentioning that when $\mu \in (0, C_H)$, an alternative definition of weak solutions was provided by Marcus and P.-T. Nguyen. In [23], a function u is a *weak solution* of (1.1) if and only if $u \in L^1(\Omega; \delta^{-\alpha-})$, $h \circ u \in L^1(\Omega; \delta^{\alpha+})$ and u satisfies (1.10) for every $\phi \in \mathbf{Y}(\Omega)$ where

$$(1.12) \quad \mathbf{Y}(\Omega) = \{ \phi \in C^2(\Omega) \text{ s.t. } \delta^{-\alpha+} \phi \in L^\infty(\Omega) \text{ and } \delta^{\alpha-} L_\mu \phi \in L^\infty(\Omega) \}.$$

The space \mathbf{Y} is larger than \mathbf{X} , however the definition in [23] is equivalent to Definition 1.2 due to the presentation form (1.11). As a consequence, as long as $\mu < C_H$, any solution u of (1.1) has $\text{tr}^*(u) = \nu$.

Convention. Throughout the paper, we assume that $\mu \in (0, \frac{1}{4}]$ and the first eigenvalue $\lambda_{\mu,1}$ of $-L_\mu$ in Ω is positive. We emphasize that if $\mu \in (0, C_H)$ then $\lambda_{\mu,1} > 0$.

For $t \in \mathbb{R}$, put

$$(1.13) \quad p_{\mu,t}^* := \frac{N + \alpha_+ + t}{N + \alpha_+ - 2}.$$

We are ready to state first existence result for (1.1) in the subcritical case.

Theorem A. Let $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha+})$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$.

(i) Assume $\gamma > -1 - \alpha_+$ and $0 < p < p_{\mu,\gamma}^*$. Then there exists a constant $C_1 > 0$ such that

$$(1.14) \quad \mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\nu])^p] \leq C_1 \mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega.$$

(ii) Assume $\gamma \in (\frac{-2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$ and $0 < p < p_{\mu,\gamma}^*$. Then there exists a constant $C_2 > 0$ such that

$$(1.15) \quad \mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p] \leq C_2 \mathbb{G}_\mu[\tau] \quad \text{a.e. in } \Omega.$$

Furthermore if $\sigma > 0$ and $\varrho > 0$ are small enough then the problem

$$(1.16) \quad \begin{cases} -L_\mu u = \delta^\gamma u^p + \sigma \tau & \text{in } \Omega, \\ u = \varrho \nu & \text{on } \partial\Omega \end{cases}$$

admits a weak solution u satisfying

$$(1.17) \quad \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] \leq u \leq C_3(\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu]) \text{ a.e. in } \Omega.$$

for some positive constant C_3 .

In addition, there exists a positive constant C_4 such that if u is a weak solution of (1.16) then

$$(1.18) \quad \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] \leq u \leq C_4(\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] + \delta^{\alpha+}) \text{ a.e. in } \Omega.$$

Moreover, if $\mu \in (0, C_H)$ then $\text{tr}^*(u) = \varrho\nu$.

In order to study (1.16) in the supercritical case, we make use of the capacities introduced in [9]. For $0 \leq \alpha \leq \beta < N$, set

$$(1.19) \quad N_{\alpha,\beta}(x, y) := \frac{1}{|x - y|^{N-\beta} \max\{|x - y|, \delta(x), \delta(y)\}^\alpha}, \quad \forall (x, y) \in \overline{\Omega} \times \overline{\Omega}$$

and

$$(1.20) \quad \mathbb{N}_{\alpha,\beta}[\tau](x) := \int_{\overline{\Omega}} N_{\alpha,\beta}(x, y) d\tau, \quad \forall \tau \in \mathfrak{M}^+(\overline{\Omega}).$$

For $a > -1$, $0 \leq \alpha \leq \beta < N$ and $s > 1$, define $\text{Cap}_{\mathbb{N}_{\alpha,\beta},s}^a$ by

$$(1.21) \quad \text{Cap}_{\mathbb{N}_{\alpha,\beta},s}^a(E) := \inf \left\{ \int_{\overline{\Omega}} \delta^a \phi^s dx : \phi \geq 0, \mathbb{N}_{\alpha,\beta}[\delta^a \phi] \geq \chi_E \right\},$$

for any Borel set $E \subset \overline{\Omega}$. For $\theta \in (0, N - 1)$ and $s > 0$, let $\text{Cap}_{\theta,s}^{\partial\Omega}$ be the capacity defined in [9, Definition 1.1]. Notice that if $\theta s > N - 1$ then $\text{Cap}_{\theta,s}^{\partial\Omega}(\{z\}) > 0$ for every $z \in \partial\Omega$.

Theorem B. *Let $\tau \in \mathfrak{M}^+(\Omega, \delta^{\alpha+})$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Assume $p > 1$ and $-2 < \gamma < -1 - \alpha_+ + p(N + \alpha_+ - 2)$. Then the following statements are equivalent.*

(i) *There exists $C > 0$ such that the following inequalities hold*

$$(1.22) \quad \tau(E) \leq C \text{Cap}_{\mathbb{N}_{2\alpha_+,2},p'}^{(p+1)\alpha_++\gamma}(E), \quad \forall \text{ Borel } E \subset \overline{\Omega}.$$

$$(1.23) \quad \nu(F) \leq C \text{Cap}_{1-\alpha_++\frac{\alpha_++\gamma+1}{p},p'}^{\partial\Omega}(F), \quad \forall \text{ Borel } F \subset \partial\Omega.$$

(ii) *There exist positive constants C_1 and C_2 such that (1.14) and (1.15) hold.*

(iii) *Problem (1.16) has a positive weak solution for $\sigma > 0$ and $\varrho > 0$ small enough.*

Remark. In the subcritical case $1 < p < p_{\mu,\gamma}^*$, $\text{Cap}_{1-\alpha_++\frac{\alpha_++\gamma+1}{p},p'}^{\partial\Omega}(\{z\}) > 0$ for any $z \in \partial\Omega$.

The second aim of the present paper is the study of weak solutions of semilinear elliptic system involving Hardy potential

$$(1.24) \quad \begin{cases} -L_\mu u = h(x, v) + \tau & \text{in } \Omega, \\ -L_\mu v = \tilde{h}(x, u) + \tilde{\tau} & \text{in } \Omega, \\ u = \nu, \quad v = \tilde{\nu} & \text{on } \partial\Omega \end{cases}$$

where $\tau, \tilde{\tau} \in \mathfrak{M}(\Omega; \delta^{\alpha+})$, $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$, $h, \tilde{h} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Definition 1.3. *A pair (u, v) is called a weak solution of (1.24) if $u \in L^1(\Omega; \delta^{\alpha+})$, $v \in L^1(\Omega; \delta^{\alpha+})$, $\tilde{h} \circ u \in L^1(\Omega; \delta^{\alpha+})$, $h \circ v \in L^1(\Omega; \delta^{\alpha+})$ and*

$$(1.25) \quad \begin{aligned} - \int_{\Omega} u L_\mu \phi dx &= \int_{\Omega} (h \circ v) \phi dx + \int_{\Omega} \phi d\tau - \int_{\Omega} \mathbb{K}_\mu[\nu] L_\mu \phi dx, \\ - \int_{\Omega} v L_\mu \phi dx &= \int_{\Omega} (\tilde{h} \circ u) \phi dx + \int_{\Omega} \phi d\tilde{\tau} - \int_{\Omega} \mathbb{K}_\mu[\tilde{\nu}] L_\mu \phi dx, \end{aligned}$$

for every $\phi \in \mathbf{X}(\Omega)$.

Elliptic systems arise in biological applications (e.g. population dynamics) or physical applications (e.g. models of nuclear reactor) and have been drawn a lot of attention (see [17, 32] and references therein). A typical case is Lane-Emden system, i.e. system (1.24) with $\mu = 0$, $h(x, v) \equiv v^p$, $\tilde{h}(x, u) \equiv u^{\tilde{p}}$. Bidaut-Véron and Yarur [10] proved various existence results for Lane-Emden system under conditions involving \mathbb{G} , \mathbb{P} and the following exponents

$$(1.26) \quad q := \tilde{p} \frac{p+1}{\tilde{p}+1}, \quad \tilde{q} := p \frac{\tilde{p}+1}{p+1}.$$

The question of solvability for (1.24) with h and \tilde{h} being more general is addressed in the present paper. We first treat the system

$$(1.27) \quad \begin{cases} -L_\mu u = \delta^\gamma v^p + \sigma\tau & \text{in } \Omega, \\ -L_\mu v = \delta^{\tilde{\gamma}} \tilde{u}^{\tilde{p}} + \tilde{\sigma}\tilde{\tau} & \text{in } \Omega, \\ u = \varrho\nu, \quad v = \tilde{\varrho}\tilde{\nu} & \text{on } \partial\Omega \end{cases}$$

where $p > 0$, $\tilde{p} > 0$, $\gamma, \tilde{\gamma} \in \mathbb{R}$, $\tau, \tilde{\tau} \in \mathfrak{M}(\Omega, \delta^{\alpha_+})$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$.

When dealing with (1.27), one encounters the following difficulties. The first one is due to the presence of the Hardy potential in the linear part of the equations. More precisely, since the singularity of the potential at the boundary is too strong, some important tools such as Hopf's lemma, the classical notion of boundary trace, etc. are invalid, and therefore the system can not be handled via classical elliptic PDEs methods. The second difficulty stems from the effect of the weights δ^γ and $\delta^{\tilde{\gamma}}$ which may vanish or blow up on the boundary. The last one comes from the interplay of the nonlinearities, the potential and measure data. The interaction between the difficulties generates an intricate dynamics both in Ω and near $\partial\Omega$ and leads to disclose new type of results.

The next theorem provides a sufficient condition for the existence of solutions of (1.27).

Theorem C. *Let $p > 0$, $\tilde{p} > 0$, $\{\tilde{\gamma}, \gamma\} \subset (-\frac{2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$, $\tau, \tilde{\tau} \in \mathfrak{M}^+(\Omega, \delta^{\alpha_+})$ and $\nu, \tilde{\nu} \in \mathfrak{M}^+(\partial\Omega)$. Assume $p\tilde{p} \neq 1$, $q < \min(q_{\mu, \gamma}^*, q_{\mu, \tilde{\gamma}}^*)$, $\mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu + \tilde{\nu}] \in L^{\tilde{p}}(\Omega, \delta^{\alpha_++\tilde{\gamma}})$. Then system (1.27) admits a weak solution (u, v) for $\sigma > 0$ and $\tilde{\sigma} > 0$ small if $p\tilde{p} > 1$, for any $\sigma > 0$ and $\tilde{\sigma} > 0$ if $p\tilde{p} < 1$. Moreover*

$$(1.28) \quad v \sim \mathbb{G}_\mu[\omega] + \mathbb{K}_\mu[\tilde{\nu}],$$

$$(1.29) \quad u \sim \mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\omega] + \mathbb{K}_\mu[\tilde{\nu}])^p] + \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu]$$

where $C = C(N, p, \tilde{p}, \mu, \Omega, \sigma, \tilde{\sigma}, \tau, \tilde{\tau})$ and

$$\omega := \delta^{\tilde{\gamma}} \left\{ \mathbb{G}_\mu[\tau + \delta^\gamma(\mathbb{K}_\mu[\tilde{\nu}])^p] \right\}^{\tilde{p}} + \delta^{\tilde{\gamma}}(\mathbb{K}_\mu[\nu])^{\tilde{p}} + \tilde{\tau}.$$

A new criterion for the existence of (1.27), expressed in terms of the capacities $\text{Cap}_{N_{\alpha, \theta, s}}^a$ and $\text{Cap}_{\theta, s}^{\partial\Omega}$, is stated in the following result.

Theorem D. *Let $p > 1$, $\tilde{p} > 1$, $\tau, \tilde{\tau} \in \mathfrak{M}^+(\Omega)$, $\nu, \tilde{\nu} \in \mathfrak{M}^+(\partial\Omega)$ and*

$$-2 < \gamma < -1 - \alpha_+ + p(N + \alpha_+ - 2) \quad \text{and} \quad -2 < \tilde{\gamma} < -1 - \alpha_+ + \tilde{p}(N + \alpha_+ - 2).$$

Assume there exists $C > 0$ such that

$$(1.30) \quad \max\{\tau(E), \tilde{\tau}(E)\} \leq C \min\{\text{Cap}_{N_{2\alpha_+, 2, p'}}^{(p+1)\alpha_++\gamma}(E), \text{Cap}_{N_{2\alpha_+, 2, \tilde{p}'}}^{(\tilde{p}+1)\alpha_++\tilde{\gamma}}(E)\}, \quad \forall E \subset \Omega,$$

$$(1.31) \quad \max\{\nu(F), \tilde{\nu}(F)\} \leq C \min\{\text{Cap}_{1-\alpha_+ + \frac{1+\alpha_++\gamma}{p}, p'}^{\partial\Omega}(F), \text{Cap}_{1-\alpha_+ + \frac{1+\alpha_++\tilde{\gamma}}{\tilde{p}}, \tilde{p}'}^{\partial\Omega}(F)\}, \quad \forall F \subset \partial\Omega.$$

Then (1.27) admits a weak solution (u, v) for $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$ small enough. Moreover (u, v) satisfies

$$(1.32) \quad \begin{aligned} \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] &\leq u \leq C(\mathbb{G}_\mu[\sigma\tau + \tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\varrho\nu + \tilde{\varrho}\tilde{\nu}]), \\ \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}] &\leq v \leq C(\mathbb{G}_\mu[\sigma\tau + \tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\varrho\nu + \tilde{\varrho}\tilde{\nu}]) \end{aligned}$$

where $C = C(N, \mu, p, \tilde{p}, \gamma, \tilde{\gamma})$. Moreover, if $\mu \in (0, C_H)$ then $\text{tr}^*(u) = \varrho\nu$ and $\text{tr}^*(v) = \tilde{\varrho}\tilde{\nu}$.

Remark. It can be inferred from Theorems A, C and D that if p and \tilde{p} are both *succritical*, i.e. $p < p_{\mu, \gamma}^*$ and $\tilde{p} < p_{\mu, \tilde{\gamma}}^*$, then system (1.27) admits a weak solutions.

Finally, we deal with elliptic systems with more general nonlinearities

$$(1.33) \quad \begin{cases} -L_\mu u = \epsilon \delta^\gamma g(v) + \sigma\tau & \text{in } \Omega, \\ -L_\mu v = \epsilon \delta^{\tilde{\gamma}} \tilde{g}(u) + \tilde{\sigma}\tilde{\tau} & \text{in } \Omega, \\ u = \varrho\nu, \quad v = \tilde{\varrho}\tilde{\nu} & \text{on } \partial\Omega. \end{cases}$$

where g and \tilde{g} are nondecreasing, continuous functions in \mathbb{R} , $\epsilon = \pm 1$, $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

We shall treat successively the cases $\epsilon = -1$ and $\epsilon = 1$. For any function f and number $t \in \mathbb{R}$, define

$$(1.34) \quad \Lambda_{f,t} := \int_1^\infty s^{-1-p_{\mu,t}^*} |f(s) - f(-s)| ds$$

with $p_{\mu,t}^*$ defined in (1.13).

Theorem E. Let $\epsilon = -1$, $\min\{\gamma, \tilde{\gamma}\} > -1 - \alpha_+$, $\sigma, \tilde{\sigma}, \varrho, \tilde{\varrho}$ be positive numbers, $\tau, \tilde{\tau} \in \mathfrak{M}(\Omega; \delta^{\alpha_+})$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$. Assume that $\Lambda_{g,\gamma} + \Lambda_{\tilde{g},\tilde{\gamma}} < \infty$ and $g(s) = \tilde{g}(s) = 0$ for any $s \leq 0$. Then system (1.33) admits a weak solution (u, v) . Moreover, if $\mu \in (0, C_H)$ then $\text{tr}^*(u) = \varrho\nu$ and $\text{tr}^*(v) = \tilde{\varrho}\tilde{\nu}$.

Put $\gamma^* := \min\{\alpha_+ + \gamma, \alpha_+ + \tilde{\gamma}, -\alpha_-\} > -1$. When $\epsilon = 1$, different phenomenon occurs, which is reflected in the following result.

Theorem F. Let $\epsilon = 1$, $\min\{\gamma, \tilde{\gamma}\} > -1 - \alpha_+$, $\tau, \tilde{\tau} \in \mathfrak{M}(\Omega; \delta^{\alpha_+})$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$.

I. SUBCRITICALITY. Assume that $\Lambda_{g,\gamma} + \Lambda_{\tilde{g},\tilde{\gamma}} < \infty$. In addition, assume that there exist $q_1 > 1$, $a_1 > 0$, $b_1 > 0$ such that

$$(1.35) \quad |g(s)| \leq a_1 |s|^{q_1} + b_1, \quad \forall s \in [-1, 1],$$

$$(1.36) \quad |\tilde{g}(s)| \leq a_1 |s|^{q_1} + b_1, \quad \forall s \in [-1, 1].$$

Then (1.33) admits a weak solution for $b_1, \sigma, \tilde{\sigma}, \varrho, \tilde{\varrho}$ small enough.

II. SUBLINEARITY. Assume that there exist $q_1 > 1$, $q_2 \in (0, 1]$, $a_2 > 0$ and $b_2 > 0$ such that $\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|] \in L^{q_1}(\Omega; \delta^{\gamma^*})$ and

$$(1.37) \quad |g(s)| \leq a_2 |s|^{q_1} + b_2, \quad \forall s \in \mathbb{R},$$

$$(1.38) \quad |\tilde{g}(s)| \leq a_2 |s|^{q_2} + b_2, \quad \forall s \in \mathbb{R}.$$

(a) If $q_1 q_2 = 1$ and $a_2 > 0$ is small then (1.33) admits a weak solution for any $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

(b) If $q_1 q_2 < 1$ then (1.33) admits a weak solution for any $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

III. SUBCRITICALITY AND SUBLINEARITY. Assume that $\Lambda_{g,\gamma} < \infty$. In addition, assume that there exist $a_1 > 0$, $a_2 > 0$, $q_1 \in (1, q_{\mu,\gamma}^*)$, $q_2 \in (0, 1]$, $b_1 > 0$, $b_2 > 0$ such that (1.35) and (1.38) hold.

- (a) If $q_1 q_2 > 1$ then (1.33) admits a weak solution for $b_1, b_2, \sigma, \tilde{\sigma}, \varrho, \tilde{\varrho}$ small enough.
- (b) If $q_2 p_{\mu,\gamma}^* = 1$ and a_2 is small enough then (1.33) admits a weak solution for any $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.
- (c) If $q_2 p_{\mu,\gamma}^* < 1$ then (1.33) admits a weak solution for every $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

If $\mu \in (0, C_H)$ then $\text{tr}^*(u) = \varrho\nu$ and $\text{tr}^*(v) = \tilde{\varrho}\tilde{\nu}$.

The paper is organized as follows. In Section 2 we establish estimates for Green kernel and Martin kernel and study properties of weak solutions of (1.1) and (1.24) in normalized boundary trace setting. Theorems A and B are proved in Section 3. Sufficient conditions for the existence of weak solutions to elliptic systems with power source terms (Theorems C and D) are presented in Section 4. Finally, in Section 5, we obtain existence results for elliptic systems with more general nonlinearities (Theorems E and F) due to Schauder fixed point theorem.

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2. PRELIMINARIES

2.1. Green kernel and Martin kernel. Denote $L_w^p(\Omega; \tau)$, $1 \leq p < \infty$, $\tau \in \mathfrak{M}^+(\Omega)$, the weak L^p space (or Marcinkiewicz space) (see [29]). When $\tau = \delta^\alpha dx$, for simplicity, we use the notation $L_w^p(\Omega; \delta^\alpha)$. Notice that, for every $\alpha > -1$,

$$(2.1) \quad L_w^p(\Omega; \delta^\alpha) \subset L^r(\Omega; \delta^\alpha), \quad \forall r \in [1, p).$$

Moreover for any $u \in L_w^p(\Omega, \delta^\alpha)$ ($\alpha > -1$),

$$(2.2) \quad \int_{\{|u| \geq s\}} \delta^\alpha dx \leq s^{-p} \|u\|_{L_w^p(\Omega; \delta^\alpha)}^p, \quad \forall s > 0.$$

Let G_μ and K_μ be the Green kernel and Martin kernel of $-L_\mu$ in Ω and by \mathbb{G}_μ and \mathbb{K}_μ be the corresponding Green operator and Martin operator, namely

$$(2.3) \quad \mathbb{G}_\mu[\tau](x) = \int_\Omega G_\mu(x, y) d\tau(y), \quad \forall \tau \in \mathfrak{M}(\Omega),$$

$$(2.4) \quad \mathbb{K}_\mu[\nu](x) = \int_{\partial\Omega} K_\mu(x, z) d\nu(z), \quad \forall \nu \in \mathfrak{M}(\partial\Omega).$$

Let us recall a result from [8] which will be useful in the sequel.

Proposition 2.1. ([8, Lemma 2.4]) *Let ω be a nonnegative bounded Radon measure in $D = \Omega$ or $\partial\Omega$ and $\eta \in C(\Omega)$ be a positive weight function. Let H be a continuous nonnegative function on $\{(x, y) \in \Omega \times D : x \neq y\}$. For any $\lambda > 0$ we set*

$$A_\lambda(y) := \{x \in \Omega \setminus \{y\} : H(x, y) > \lambda\},$$

$$m_\lambda(y) := \int_{A_\lambda(y)} \eta(x) dx.$$

Suppose that there exist $C > 0$ and $k > 1$ such that $m_\lambda(y) \leq C\lambda^{-k}$ for every $\lambda > 0$. Then the operator

$$\mathbb{H}[\omega](x) := \int_D H(x, y) d\omega(y)$$

belongs to $L_w^k(\Omega; \eta dx)$ and

$$\|\mathbb{H}[\omega]\|_{L_w^k(\Omega; \eta dx)} \leq (1 + \frac{Ck}{k-1})\omega(D).$$

By combining (2.3), (2.4) and the above Lemma we have the following result.

Lemma 2.2. *Let*

$$\beta \in \left(-\frac{N\alpha_+}{N+2\alpha_+-2}, \frac{\alpha_+N}{N-2} \right).$$

Then there exists a positive constant $C = C(N, \mu, \beta, \Omega)$ such that

$$(2.5) \quad \sup_{\xi \in \Omega} \left\| \frac{G_\mu(\cdot, \xi)}{\delta(\xi)^{\alpha_+}} \right\|_{L_w^{\frac{N+\beta}{N+\alpha_+-2}}(\Omega; \delta^\beta)} < C.$$

Proof. Let $\xi \in \Omega$. We will apply Proposition 2.1 with $D = \Omega$, $\eta = \delta^\beta$ with $\beta > -1$, $\omega = \delta^{\alpha_+} \delta_\xi$, where δ_ξ is the Dirac measure concentrated at ξ , and

$$H(x, y) = \frac{G_\mu(x, y)}{\delta(y)^{\alpha_+}}.$$

Then

$$\mathbb{H}[\omega](x) = \int_\Omega \frac{G_\mu(x, y)}{\delta(y)^{\alpha_+}} \delta(y)^{\alpha_+} d\delta_\xi(y) = G_\mu(x, \xi).$$

From (2.3), there exists $C = C(N, \mu, \Omega)$ such that, for every $(x, y) \in \Omega \times \Omega$, $x \neq y$,

$$(2.6) \quad G_\mu(x, y) \leq C\delta(y)^{\alpha_+} |x - y|^{2-N-\alpha_+},$$

$$(2.7) \quad G_\mu(x, y) \leq C \frac{\delta(y)^{\alpha_+}}{\delta(x)^{\alpha_+}} |x - y|^{2-N},$$

$$(2.8) \quad G_\mu(x, y) \leq C\delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x - y|^{2-N-2\alpha_+}.$$

By (2.6), for any $x \in A_\lambda(y)$,

$$(2.9) \quad \lambda \leq C|x - y|^{2-N-\alpha_+},$$

and from (2.7) and (2.8)

$$(2.10) \quad \delta(x)^{\alpha_+} \leq \frac{C}{\lambda} |x - y|^{2-N} \quad \text{and} \quad \delta(x)^{\alpha_+} \geq C\lambda |x - y|^{N+2\alpha_+-2}$$

We consider two cases: $\beta \geq 0$ and $-1 < \beta < 0$.

Case 1: $\beta \geq 0$. Due to (2.9) and (2.10) we have

$$\begin{aligned} m_\lambda(y) &= \int_{A_\lambda(y)} \delta(x)^\beta dx \leq \int_{A_\lambda(y)} \left(\frac{C}{\lambda} |x - y|^{2-N} \right)^{\frac{\beta}{\alpha_+}} dx \\ &\leq C\lambda^{-\frac{N+\beta}{N+\alpha_+-2}}, \end{aligned}$$

with $\beta < \frac{N\alpha_+}{N-2}$. Observe that $\omega(\Omega) = \delta(\xi)^{\alpha_+}$, by Proposition 2.1, we get

$$\left\| \frac{G_\mu(\cdot, \xi)}{\delta(\xi)^{\alpha_+}} \right\|_{L_w^{\frac{N+\beta}{N+\alpha_+-2}}(\Omega; \delta^\beta)} \leq C\delta(\xi)^{\alpha_+}.$$

This implies (2.5).

Case 2: $-1 < \beta < 0$. By (2.9) and (2.10) we have

$$\begin{aligned} m_\lambda(y) &= \int_{A_\lambda(y)} \delta^\beta(x) dx \leq \int_{A_\lambda(y)} (C\lambda|x-y|^{N+2\alpha_+-2})^{\frac{\beta}{\alpha_+}} dx \\ &\leq C\lambda^{-\frac{N+\beta}{N+\alpha_+-2}}, \end{aligned}$$

with $\beta > -\frac{N\alpha_+}{N+2\alpha_+-2}$. By arguing similarly as in Case 1, we get (2.5). \square

Lemma 2.3. *Let $\beta > -1$. Then there exists a positive constant $C = C(N, \mu, \beta, \Omega)$ such that*

$$\sup_{\xi \in \partial\Omega} \|K_\mu(\cdot, \xi)\|_{L_w^{\frac{N+\beta}{N+\alpha_+-2}}(\Omega; \delta^\beta)} < C.$$

Proof. Let $\xi \in \partial\Omega$. We will apply Proposition 2.1 with $D = \partial\Omega$, $\eta = \delta^\beta$ with $\beta > -1$ and $\omega = \delta_\xi$. The rest of the proof can be proceeded as in the proof of Lemma 2.2 and we omit it. \square

In view of (2.1), Lemma 2.2 and Lemma 2.3, one can obtain easily the following proposition (see also [23, 31]).

Proposition 2.4. (i) *Let $\beta \in (-\frac{N\alpha_+}{N+2\alpha_+-2}, \frac{\alpha_+N}{N-2})$. Then there exists a constant $c = c(N, \mu, \beta, \Omega)$ such that*

$$(2.11) \quad \|\mathbb{G}_\mu[\tau]\|_{L_w^{\frac{N+\beta}{N+\alpha_+-2}}(\Omega, \delta^\beta)} \leq c \|\tau\|_{\mathfrak{M}(\Omega; \delta^{\alpha_+})}, \quad \forall \tau \in \mathfrak{M}(\Omega; \delta^{\alpha_+}).$$

(ii) *Let $\beta > -1$. Then there exists a constant $c = c(N, \mu, \beta, \Omega)$ such that*

$$(2.12) \quad \|\mathbb{K}_\mu[\nu]\|_{L_w^{\frac{N+\beta}{N+\alpha_+-2}}(\Omega, \delta^\beta)} \leq c \|\nu\|_{\mathfrak{M}(\partial\Omega)}, \quad \forall \nu \in \mathfrak{M}(\partial\Omega).$$

2.2. Normalized boundary trace. In this subsection, we present results concerning elliptic equations and systems with $\mu \in (0, C_H)$. We recall that the definition of normalized boundary trace is given in Definition 1.1.

Definition 2.5. (i) *Let $\tau \in \mathfrak{M}(\Omega; \delta^{\alpha_+})$. A function u is a distributional solution of*

$$(2.13) \quad -L_\mu u = \tau$$

if $u \in L_{loc}^1(\Omega)$ and (2.13) is understood in the sense of distributions.

(ii) *Let $\tau \in \mathfrak{M}(\Omega; \delta^{\alpha_+})$ and $\nu \in \mathfrak{M}(\partial\Omega)$. A function u is a very weak solution of*

$$(2.14) \quad -L_\mu u = \tau \quad \text{in } \Omega, \quad \text{tr}^*(u) = \nu,$$

if u is a solution of (2.13) and u admits normalized boundary trace ν .

Various properties of solutions of (2.13) and (2.14) were obtained in [23, Proposition I]) and are recalled in the following proposition.

Proposition 2.6. (i) *If $\tau = 0$ then problem (2.14) has a unique very weak solution, $u = \mathbb{K}_\mu[\nu]$. If u is an L_μ -harmonic function and $\text{tr}^*(u) = 0$ then $u = 0$.*

(ii) *If $\tau \in \mathfrak{M}(\Omega; \delta^{\alpha_+})$ then $\mathbb{G}_\mu[\tau]$ has normalized trace zero. Thus $\mathbb{G}_\mu[\tau]$ is a solution of (2.14) with $\nu = 0$.*

(iii) For every $\nu \in \mathfrak{M}(\partial\Omega)$ and $\tau \in \mathfrak{M}(\Omega; \delta^{\alpha+})$, problem (2.14) has a unique very weak solution. The solution is given by

$$(2.15) \quad u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu].$$

Moreover, there exists a positive constant $c = c(N, \mu, \Omega)$ such that

$$(2.16) \quad \|u\|_{L^1(\Omega; \delta^{-\alpha-})} \leq c(\|\tau\|_{\mathfrak{M}(\Omega; \delta^{\alpha+})} + \|\nu\|_{\mathfrak{M}(\partial\Omega)}).$$

(iv) u is a very weak solution of (2.14) if and only if $u \in L^1(\Omega; \delta^{-\alpha-})$ and

$$(2.17) \quad -\int_{\Omega} u L_\mu \zeta dx = \int_{\Omega} \zeta d\tau - \int_{\Omega} \mathbb{K}_\mu^\Omega[\nu] L_\mu \zeta dx, \quad \forall \zeta \in \mathbf{Y}(\Omega)$$

where $\mathbf{Y}(\Omega)$ is given in (1.12).

Proposition 2.6 enables to derive properties of solutions of (1.1), as well as (1.24).

Proposition 2.7. Let $\tau \in \mathfrak{M}(\Omega; \delta^{\alpha+})$ and $\nu \in \mathfrak{M}(\partial\Omega)$. The following statements are equivalent.

- (i) u is a weak solution of (1.1).
- (ii) $h \circ u \in L^1(\Omega; \delta^{\alpha+})$ and (1.1) holds.
- (iii) $u \in L^1(\Omega; \delta^{-\alpha-})$, $h \circ u \in L^1(\Omega; \delta^{\alpha+})$ and (1.10) holds for every $\phi \in \mathbf{Y}(\Omega)$.
- (iv) u is a solution of

$$(2.18) \quad -L_\mu u = h \circ u + \tau \quad \text{in } \Omega$$

in the sense of distributions and $\text{tr}^*(u) = \nu$.

Proof. We infer from [18] that (i) \iff (ii). By an argument similar to that of the proof of [31, Theorem A], we deduce that (ii) \iff (iii) \iff (iv). \square

For $\beta > 0$, put

$$(2.19) \quad \Omega_\beta := \{x \in \Omega : \delta(x) < \beta\}, \quad D_\beta := \{x \in \Omega : \delta(x) > \beta\}, \quad \Sigma_\beta := \{x \in \Omega : \delta(x) = \beta\}.$$

Lemma 2.8. There exists $\beta_* > 0$ such that for every point $x \in \overline{\Omega}_{\beta_*}$, there exists a unique point $\sigma_x \in \partial\Omega$ such that $x = \sigma_x - \delta(x)\mathbf{n}_{\sigma_x}$. The mappings $x \mapsto \delta(x)$ and $x \mapsto \sigma_x$ belong to $C^2(\overline{\Omega}_{\beta_*})$ and $C^1(\overline{\Omega}_{\beta_*})$ respectively. Moreover, $\lim_{x \rightarrow \sigma(x)} \nabla \delta(x) = -\mathbf{n}_{\sigma_x}$.

A version of the above result is valid for elliptic systems.

Proposition 2.9. Let $\tau, \tilde{\tau} \in \mathfrak{M}(\Omega; \delta^{\alpha+})$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$. Then the following statements are equivalent.

- (i) (u, v) is a weak solution of (1.24).
- (ii) $\tilde{h} \circ u \in L^1(\Omega; \delta^{\alpha+})$, $h \circ v \in L^1(\Omega; \delta^{\alpha+})$ and

$$(2.20) \quad u = \mathbb{G}_\mu[h \circ v] + \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu], \quad v = \mathbb{G}_\mu[\tilde{h} \circ u] + \mathbb{G}_\mu[\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\nu}].$$

(iii) $u \in L^1(\Omega; \delta^{-\alpha-})$, $v \in L^1(\Omega; \delta^{-\alpha-})$, $\tilde{h} \circ u \in L^1(\Omega; \delta^{\alpha+})$, $h \circ v \in L^1(\Omega; \delta^{\alpha+})$ and (1.25) holds for every $\phi \in \mathbf{Y}(\Omega)$.

- (iv) (u, v) is a solution of

$$(2.21) \quad \begin{cases} -L_\mu u = h \circ v + \tau & \text{in } \Omega, \\ -L_\mu v = \tilde{h} \circ u + \tilde{\tau} & \text{in } \Omega, \end{cases}$$

in the sense of distributions and $\text{tr}^*(u) = \nu$ and $\text{tr}^*(v) = \tilde{\nu}$.

Proof. Due to [18], (i) \iff (ii). We next prove other implications.

(iv) \implies (ii). Assume (u, v) is a weak solution of (1.24). Put $\omega := h \circ v$ and denote $\omega_\beta := \omega|_{D_\beta}$, $\tau_\beta := \tau|_{D_\beta}$ and $\lambda_\beta := u|_{\Sigma_\beta}$ for $\beta \in (0, \beta_*)$. Consider the boundary value problem

$$-L_\mu w = \omega_\beta + \tau_\beta \quad \text{in } D_\beta, \quad w = \lambda_\beta \quad \text{on } \Sigma_\beta.$$

This problem admits a unique solution w_β (the uniqueness is derived from [7, Lemma 2.1] since $\mu < C_H(\Omega)$). Therefore $w_\beta = u|_{D_\beta}$. We have

$$u|_{D_\beta} = w_\beta = \mathbb{G}_\mu^{D_\beta}[\omega_\beta] + \mathbb{G}_\mu^{D_\beta}[\tau_\beta] + \mathbb{P}_\mu^{D_\beta}[\lambda_\beta]$$

where $\mathbb{G}_\mu^{D_\beta}$ and $\mathbb{P}_\mu^{D_\beta}$ are respectively Green kernel and Poisson kernel of $-L_\mu$ in D_β .

It follows that

$$\left| \int_{D_\beta} \mathbb{G}_\mu^{D_\beta}(\cdot, y)(h \circ v)(y) dy \right| = \left| \mathbb{G}_\mu^{D_\beta}[\tau_\beta] \right| \leq |u|_{D_\beta} + |\mathbb{G}_\mu^\Omega[\tau]| + |\mathbb{K}_\mu^\Omega[\nu]|.$$

Letting $\beta \rightarrow 0$, we get

$$(2.22) \quad \left| \int_\Omega G_\mu(\cdot, y)(h \circ v)(y) dy \right| < \infty.$$

Fix a point $x_0 \in \Omega$. Keeping in mind that $G_\mu(x_0, y) \sim \delta(y)^{\alpha_+}$ for every $y \in \Omega_{\beta_*}$, we deduce from (2.22) that $h \circ v \in L^1(\Omega; \delta^{\alpha_+})$. Similarly, one can show that $\tilde{h} \circ u \in L^1(\Omega; \delta^{\alpha_+})$. Thanks to Proposition 2.6 (v), we obtain (2.20).

(ii) \implies (iv). Assume u and v are functions such that $\tilde{h} \circ u \in L^1(\Omega; \delta^{\alpha_+})$, $h \circ v \in L^1(\Omega; \delta^{\alpha_+})$ and (2.20) holds. By Proposition 2.6 (i) $L_\mu \mathbb{K}_\mu[\nu] = L_\mu \mathbb{K}_\mu[\tilde{\nu}] = 0$, which implies that (u, v) is a solution of (2.21). On the other hand, since $\tilde{h} \circ u \in L^1(\Omega; \delta^{\alpha_+})$ and $h \circ v \in L^1(\Omega; \delta^{\alpha_+})$, we deduce from Proposition 2.6 (ii) that $\text{tr}^*(\mathbb{G}_\mu[\tilde{h} \circ u]) = \text{tr}^*(\mathbb{G}_\mu[h \circ v]) = 0$. Consequently, $\text{tr}^*(u) = \text{tr}^*(\mathbb{K}_\mu[\nu]) = \nu$ and $\text{tr}^*(v) = \text{tr}^*(\mathbb{K}_\mu[\tilde{\nu}]) = \tilde{\nu}$.

(iv) \implies (iii). Assume (u, v) is a positive solution of (2.21) in the sense of distributions. From the implication (iv) \implies (ii), we deduce that $u \in L^1(\Omega; \delta^{-\alpha_-})$, $v \in L^1(\Omega; \delta^{-\alpha_-})$, $\tilde{h} \circ u \in L^1(\Omega; \delta^{\alpha_+})$ and $h \circ v \in L^1(\Omega; \delta^{\alpha_+})$. Hence, by Proposition 2.6 (vi), (1.25) holds for every $\phi \in \mathbf{Y}(\Omega)$.

(iii) \implies (iv). This implication follows straightfoward from Proposition 2.6 (vi). \square

3. THE SCALAR PROBLEM

In what follows the notation $f_1 \sim f_2$ means: there exists a positive constant c such that $c^{-1}f_1 \leq f_2 \leq cf_1$ in the domain of the two functions or in a specified subset of this domain. In the latter case, the constant depends on the subset.

3.1. Concavity properties and Green properties. Here we give some concavity lemmas that will be employed in the sequel.

Proposition 3.1. *Let $\gamma > -1 - \alpha_+$ and $\varphi \in L^1(\Omega, \delta^{\alpha_+})$, $\varphi \geq 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$. Set*

$$w := \mathbb{G}_\mu[\varphi + \tau] \quad \text{and} \quad \psi = \mathbb{G}_\mu[\tau].$$

Let ϕ be a concave nondecreasing C^2 function on $[0, \infty)$, such that $\phi(1) \geq 0$. Then $\phi'(w/\psi)\varphi \in L^1(\Omega; \delta^{\alpha_++\gamma})$ and

$$-L_\mu(\psi\phi(w/\psi)) \geq \phi'(w/\psi)\varphi,$$

in the weak sense in Ω .

Proof. Let $\{\varphi_n\}, \tau_n \in C^\infty(\overline{\Omega})$ such that $\varphi_n \rightarrow \varphi$ in $L^1(\Omega, \delta^{\alpha+})$ and $\tau_n \rightarrow \tau$. Set

$$w_n = \mathbb{G}_\mu[\varphi_n + \tau_n] \quad \text{and} \quad \psi_n = \mathbb{G}_\mu[\tau_n].$$

Since $w_n \geq \psi_n > 0$ for any $n \geq n_0$ for some $n_0 \in \mathbb{N}$, we have by straightforward calculations

$$\begin{aligned} -\Delta(\psi_n \phi(\frac{w_n}{\psi_n})) &= (-\Delta\psi_n) \left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n} \phi'(\frac{w_n}{\psi_n}) \right) + (-\Delta w_n) \phi'(\frac{w_n}{\psi_n}) \\ &\quad - \psi_n \phi''(\frac{w_n}{\psi_n}) \left| \nabla \left(\frac{w_n}{\psi_n} \right) \right|^2. \end{aligned}$$

Now note that, since $\phi' \geq 0$, we have

$$(-\Delta w_n) \phi'(\frac{w_n}{\psi_n}) \geq \phi'(\frac{w_n}{\psi_n}) \left(-\Delta\psi_n - \mu \frac{\psi_n}{\delta^2} + \mu \frac{w_n}{\delta^2} + \varphi_n \right).$$

This, together with the fact that $\phi(t) - t\phi'(t) + \phi'(t) \geq 0$ for any $t \geq 1$, implies

$$\begin{aligned} &(-\Delta\psi_n) \left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n} \phi'(\frac{w_n}{\psi_n}) \right) + (-\Delta w_n) \phi'(\frac{w_n}{\psi_n}) \\ &\geq (-\Delta\psi_n) \left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n} \phi'(\frac{w_n}{\psi_n}) + \phi'(\frac{w_n}{\psi_n}) \right) + \phi'(\frac{w_n}{\psi_n}) \left(-\mu \frac{\psi_n}{\delta^2} + \mu \frac{w_n}{\delta^2} + \varphi_n \right) \\ &\geq \mu \frac{\psi_n}{\delta^2} \left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n} \phi'(\frac{w_n}{\psi_n}) + \phi'(\frac{w_n}{\psi_n}) \right) + \phi'(\frac{w_n}{\psi_n}) \left(-\mu \frac{\psi_n}{\delta^2} + \mu \frac{w_n}{\delta^2} + \varphi_n \right) \\ &= \frac{\mu}{\delta^2} \psi_n \phi(\frac{w_n}{\psi_n}) + \phi'(\frac{w_n}{\psi_n}) \varphi_n. \end{aligned}$$

Thus we have proved

$$-L_\mu(\psi_n \phi(\frac{w_n}{\psi_n})) \geq \phi'(\frac{w_n}{\psi_n}) \varphi_n.$$

Also

$$\psi_n \phi(\frac{w_n}{\psi_n}) \leq \psi_n (\phi(0) + \phi'(0) \frac{w_n}{\psi_n}) \leq C(\psi_n + w_n)$$

and

$$-\int_{\Omega} (\psi_n \phi(\frac{w_n}{\psi_n})) L_\mu \xi \, dx \geq \int_{\Omega} \phi'(\frac{w_n}{\psi_n}) \varphi_n \xi \, dx, \quad \forall \xi \in \mathbf{X}(\Omega).$$

By passing to the limit with Lebesgues theorem and Fatous lemma, we complete the proof. \square

In the next Lemma we will prove the 3-G inequality which will be useful later.

Lemma 3.2. *There exists a positive constant $C = C(N, \mu, \Omega)$ such that*

$$(3.1) \quad \frac{G_\mu(x, y) G_\mu(y, z)}{G_\mu(x, z)} \leq C \left(\frac{\delta(y)^{\alpha+}}{\delta(x)^{\alpha+}} G_\mu(x, y) + \frac{\delta(y)^{\alpha+}}{\delta(z)^{\alpha+}} G_\mu(y, z) \right), \quad \forall (x, y, z) \in \Omega \times \Omega \times \Omega.$$

Proof. It follows from (2.3) and the inequality $|\delta(x) - \delta(y)| \leq |x - y|$ that

$$\begin{aligned} G_\mu(x, y) &\sim \min \left\{ |x - y|^{2-N}, \delta(x)^{\alpha+} \delta(y)^{\alpha+} |x - y|^{2-2\alpha+-N} \right\} \\ &\sim |x - y|^{2-N} \delta(x)^{\alpha+} \delta(y)^{\alpha+} \left(\max \left\{ \delta(x)^{\alpha+} \delta(y)^{\alpha+}, |x - y|^{2\alpha+} \right\} \right)^{-1} \\ &\sim |x - y|^{2-N} \delta(x)^{\alpha+} \delta(y)^{\alpha+} (\max \{ \delta(x), \delta(y), |x - y| \})^{-2\alpha+} \\ &= \delta(x)^{\alpha+} \delta(y)^{\alpha+} N_{2\alpha+, 2}(x, y), \quad \forall x, y \in \Omega, x \neq y, \end{aligned}$$

where $N_{2\alpha_+,2}(x, y)$ is defined in (1.20) with $\alpha = 2\alpha_+$ and $\beta = 2$. By Proposition 3.13 we deduce that there exists a positive constant $C = C(N, \mu, \Omega)$ such that

$$(3.2) \quad \frac{1}{N_{2\alpha_+,2}(x, y)} \leq C \left(\frac{1}{N_{2\alpha_+,2}(x, z)} + \frac{1}{N_{2\alpha_+,2}(y, z)} \right).$$

Combining (3.2) and (3.1) implies (3.1). \square

Lemma 3.3. *Let $\gamma \in (\frac{-2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$, $0 < p < p_{\mu,\gamma}^*$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$. Then there exists constant $C_2 > 0$ such that (1.14) holds.*

Proof. First we assume that $p > 1$. By (2.11) we have that $\delta^\gamma(\mathbb{G}_\mu[\tau])^p \in L^1(\Omega, \delta^{\alpha_+})$. We write

$$\mathbb{G}_\mu[\tau](y) = \int_\Omega G_\mu(y, z) d\tau(z) = \int_\Omega \frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \delta(z)^{\alpha_+} d\tau(z),$$

thus

$$\delta(y)^\gamma (\mathbb{G}_\mu[\tau](y))^p \leq C(N, \mu, p, \tau, \Omega) \delta(y)^\gamma \int_\Omega \delta(z)^{\alpha_+} \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^p d\tau(z).$$

Consequently,

$$(3.3) \quad \mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p](x) \leq C \int_\Omega \int_\Omega \delta(y)^\gamma G_\mu(x, y) (G_\mu(y, z))^p \delta(z)^{\alpha_+(1-p)} d\tau(z) dy.$$

Also by (3.1) we obtain

$$\begin{aligned} & \int_\Omega \int_\Omega \delta(y)^\gamma G_\mu(x, y) (G_\mu(y, z))^p \delta(z)^{\alpha_+(1-p)} d\tau(z) dy \\ & \leq C \int_\Omega G_\mu(x, z) \int_\Omega \delta(y)^{\gamma+\alpha_+} \left(\left(\frac{G_\mu(x, y)}{\delta(x)^{\alpha_+}} \right) \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^{p-1} + \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^p \right) dy d\tau(z) \\ (3.4) \quad & \leq C \int_\Omega G_\mu(x, z) \int_\Omega \delta(y)^{\gamma+\alpha_+} \left(\left(\frac{G_\mu(x, y)}{\delta(x)^{\alpha_+}} \right)^p + \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^p \right) dy d\tau(z), \end{aligned}$$

where in the last inequality we have used the Hölder inequality. By (3.3), (3.4) and Lemma 2.2 we derive that

$$\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p](x) \leq C \int_\Omega G_\mu(x, z) d\tau(z).$$

Note that the above argument is still valid for $p = 1$.

If $0 \leq p < 1$ then

$$\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p] \leq C(\mathbb{G}_\mu[\delta^\gamma] + \mathbb{G}_\mu[\delta^\gamma \mathbb{G}_\mu[\tau]]).$$

Now let $0 < \varepsilon < 1$, then the function $\zeta := \delta^{\alpha_+}(1 - \delta^\varepsilon)$ satisfies

$$-L_\mu \zeta - \delta^\gamma > 0, \quad \forall x \in \Omega_\beta,$$

for $\beta > 0$ small enough, where Ω_β is defined in (2.19). Thus by a comparison argument we deduce that

$$(3.5) \quad \mathbb{G}_\mu[\delta^\gamma](x) \leq C\zeta(x), \quad \forall x \in \Omega_\beta.$$

Choose now $x_0 \in \Omega$, $r = \frac{\delta(x_0)}{2}$ and set $\tau_{B_r(x_0)} = \chi_{B_r(x_0)} \tau$. Then we get

$$(3.6) \quad \mathbb{G}_\mu[\tau](x) \geq C\mathbb{G}_\mu[\tau_{B_r(x_0)}](x) \geq C\delta(x)^{\alpha_+}, \quad \forall x \in \Omega \setminus B_r(x_0).$$

Thus (1.14) follows from (3.5) and (3.6). \square

Actually (1.14) is a sufficient condition for the existence of weak solution of

$$(3.7) \quad \begin{cases} -L_\mu u = \delta^\gamma u^p + \sigma\tau & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 3.4. *Let $\gamma \in (\frac{-2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$, $0 < p \neq 1$, $\sigma > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$. Assume that there exists a positive constant C_2 such that (1.14) holds. Then problem (3.7) admits a solution u satisfying*

$$(3.8) \quad \mathbb{G}_\mu[\sigma\tau] \leq u \leq C'\mathbb{G}_\mu[\sigma\tau] \quad \text{a. e. in } \Omega,$$

with a constant $C' > 0$, for any $\sigma > 0$ small enough if $p > 1$, for any $\sigma > 0$ if $p < 1$.

Proof. The proof is similar to that of [10, Theorem 3.4] with obvious modifications; hence we omit it. \square

Estimate (1.14) is also a necessary condition for the existence of weak solution of (3.7).

Proposition 3.5. *Let $\gamma \in (\frac{-2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$, $p > 1$, $\sigma > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$. Assume that problem (3.7) admits a solution. Then (1.14) holds with $C_2 = \frac{1}{p-1}$.*

Proof. By adapting the argument used in the proof of [10, Proposition 3.5], together with Proposition 3.1, one can get the desired result. \square

Proposition 3.6. *Let $\gamma \in (\frac{-2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$ and $0 < p < p_{\mu,\gamma}^*$. Assume that w is a solution of (3.7). Then there exists a positive constant $C = C(N, \mu, \Omega, \sigma, \tau)$ such that*

$$(3.9) \quad \mathbb{G}_\mu[\sigma\tau] \leq w \leq C(\mathbb{G}_\mu[\sigma\tau] + \delta^{\alpha_+}) \quad \text{a.e. in } \Omega.$$

Proof. We follow the idea in the proof of [10, Theorem 3.6]. We may assume that $\sigma = 1$. If $0 \leq p < 1$, then

$$w = \mathbb{G}_\mu[\delta^\gamma w^p + \tau] \leq C(\mathbb{G}_\mu[\delta^\gamma + \tau] + \mathbb{G}_\mu[\delta^\gamma w]).$$

On the other hand, from (3.5), we get $\mathbb{G}_\mu[\delta^\gamma] \leq C\delta^{\alpha_+}$ a.e. in Ω . As a consequence

$$w \leq C(\mathbb{G}_\mu[\delta^\gamma w] + \mathbb{G}_\mu[\tau] + \delta^{\alpha_+}) \quad \text{a.e. in } \Omega.$$

Therefore it is sufficient to deal with the term $\mathbb{G}_\mu[\delta^\gamma w]$ and we may assume that $p \geq 1$. Set

$$w_1 := w - \mathbb{G}_\mu[\tau] = \mathbb{G}_\mu[\delta^\gamma w^p],$$

hence $w = w_1 + \mathbb{G}_\mu[\tau]$. Since $w \in L^p(\Omega; \delta^{\alpha_++\gamma})$ (by assumption), it follows that $\delta^\gamma w + \tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$, therefore by (2.11), $w \in L^s(\Omega; \delta^{\alpha_++\gamma})$, for all $1 \leq s < p_{\mu,\gamma}^*$. Thus there exists $k_0 > 1$ such that $w^p \in L^{k_0}(\Omega; \delta^{\alpha_++\gamma})$.

Let $p < s < p_{\mu,\gamma}^*$. By Hölder inequality we obtain

$$\begin{aligned} w_1(x)^{k_0s} &= (\mathbb{G}_\mu[\delta^\gamma w^p](x))^{k_0s} = \left(\int_\Omega \frac{G_\mu(x, y)}{\delta(y)^{\alpha_+}} \delta(y)^{\gamma+\alpha_+} w(y)^p dy \right)^{k_0s} \\ &\leq \left(\int_\Omega \frac{G_\mu(x, y)}{\delta(y)^{\alpha_+}} \delta(y)^{\gamma+\alpha_+} w(y)^{k_0p} dy \right)^s \left(\int_\Omega \frac{G_\mu(x, y)}{\delta(y)^{\alpha_+}} \delta(y)^{\gamma+\alpha_+} dy \right)^{\frac{(k_0-1)s}{k_0}} \\ &\leq C \int_\Omega \left(\frac{G_\mu(x, y)}{\delta(y)^{\alpha_+}} \right)^s \delta(y)^{\gamma+\alpha_+} w(y)^{k_0p} dy. \end{aligned}$$

This, joint with Lemma 2.2, yields

$$\int_\Omega w_1(y)^{k_0s} \delta(y)^{\gamma+\alpha_+} dy \leq C \int_\Omega w(y)^{k_0p} \delta(y)^{\gamma+\alpha_+} \int_\Omega \left(\frac{G_\mu(x, y)}{\delta(y)^{\alpha_+}} \right)^s \delta(x)^{\gamma+\alpha_+} dx dy < c.$$

Since $w^p \leq C(p)(w_1^p + (\mathbb{G}_\mu[\tau])^p)$, by Lemma 3.3 we have

$$w \leq C(\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p] + w_2) + \mathbb{G}_\mu[\tau] \leq C(\mathbb{G}_\mu[\tau] + w_2),$$

where $w_2 := \mathbb{G}_\mu[\delta^\gamma w_1^p]$. Note that $w_2 \in L^{\frac{k_0 s^2}{p^2}}(\Omega; \delta^{\gamma+\alpha_+})$.

By induction we define $w_n := \mathbb{G}_\mu[\delta^\gamma w_{n-1}^p]$ and we have

$$w \leq C(\mathbb{G}_\mu[\tau] + w_n)$$

and

$$w_n^p \in L^{s_n}(\Omega; \delta^{\gamma+\alpha_+}), \quad s_n = \frac{k_0 s^n}{p^n}.$$

Since $s_n \rightarrow \infty$, by [29, Lemma 2.3.2] we have for $1 < s < p_{\mu, \gamma}^*$,

$$\begin{aligned} w_n &= \mathbb{G}_\mu[\delta^\gamma w_{n-1}^p] \\ &\leq C \int_{\Omega} |x-y|^{2-\alpha_+-N} w_{n-1}^p \delta(y)^{\gamma+\alpha_+} dy \\ &\leq C \left(\int_{\Omega} |x-y|^{(2-\alpha_+-N)s} \delta(y)^{\gamma+\alpha_+} dy + \int_{\Omega} |x-y|^{2-\alpha_+-N} w_{n-1}^{\frac{ps}{s-1}} \delta(y)^{\gamma+\alpha_+} dy \right) \\ &\leq C', \end{aligned}$$

for n large enough. Therefore we obtain

$$w \leq C(\mathbb{G}_\mu[\tau] + 1),$$

which implies, with another $C > 0$,

$$w \leq C(\mathbb{G}_\mu[\tau] + \mathbb{G}_\mu[\delta^\gamma]).$$

This, together with (3.5), implies (3.9) for some constant $C > 0$ depending on $\|w\|_{L^p(\Omega, \delta^{\gamma+\alpha_+})}$.

Let ζ be the unique solution of

$$\begin{cases} -L_\mu \zeta = \delta^{\gamma+\alpha_+} & \text{in } \Omega \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

In view of the proof of (3.5) we deduce that $\zeta \sim \delta^{\alpha_+}$. Since w is a weak solution of (3.7), by taking ζ as a test function, we get

$$-\int_{\Omega} w L_\mu \zeta dx = \int_{\Omega} w \delta^{\gamma+\alpha_+} dx = \int_{\Omega} \delta^\gamma w^p \zeta dx + \int_{\Omega} \zeta d\tau.$$

By Hölder inequality we can easily conclude that the constant C in (3.9) does not depend on $\|w\|_{L^p(\Omega, \delta^{\gamma+\alpha_+})}$. \square

3.2. New Green properties.

Lemma 3.7. *Let $\gamma \in (\frac{-2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$, $0 < p < p_{\mu, \gamma}^*$, $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$ and s such that*

$$(3.10) \quad \max\left(0, p - \frac{2+\gamma}{N+\alpha_+-2}\right) < s \leq 1.$$

Then there exists constant $C > 0$ such that

$$(3.11) \quad \mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p] \leq C(\mathbb{G}_\mu[\tau])^s \quad \text{a.e. in } \Omega.$$

Proof. First we assume that $p > 1$. In view of the proof of Lemma 3.3, we have

$$\begin{aligned}
\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p](x) &\leq C \int_\Omega \int_\Omega \delta(y)^\gamma G_\mu(x, y) ((G_\mu(y, z))^p \delta(z)^{\alpha_+(1-p)} d\tau(z) dy \\
&= C \int_\Omega \int_\Omega \delta(y)^\gamma (G_\mu(x, y))^{1-s} (G_\mu(x, y))^s (G_\mu(y, z))^s \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^{p-s} \delta(z)^{\alpha_+(1-s)} d\tau(z) dy \\
(3.12) \quad &\leq C \int_\Omega (G_\mu(x, z))^s \delta(z)^{\alpha_+(1-s)} \int_\Omega \delta(y)^{\gamma+\alpha_+} \frac{G_\mu(x, y)}{\delta(x)} \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^{p-s} dy d\tau(z) \\
(3.13) \quad &+ C \int_\Omega (G_\mu(x, z))^s \delta(z)^{\alpha_+(1-s)} \int_\Omega \delta(y)^{\gamma+\alpha_+} \left(\frac{G_\mu(x, y)}{\delta(x)} \right)^{1-s} \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^p dy d\tau(z) \\
(3.14) \quad &\leq C \int_\Omega (G_\mu(x, z))^s \delta(z)^{\alpha_+(1-s)} \int_\Omega \delta(y)^{\gamma+\alpha_+} \left(\frac{G_\mu(x, y)}{\delta(x)} \right)^{p-s+1} dy d\tau(z) \\
(3.15) \quad &+ \int_\Omega (G_\mu(x, z))^s \delta(z)^{\alpha_+(1-s)} \int_\Omega \delta(y)^{\gamma+\alpha_+} \left(\frac{G_\mu(y, z)}{\delta(z)^{\alpha_+}} \right)^{p-s+1} dy d\tau(z) \\
(3.16) \quad &\leq C \int_\Omega \left(\frac{G_\mu(x, z)}{\delta(z)^{\alpha_+}} \right)^s \delta(z)^{\alpha_+} d\tau(z) \\
(3.17) \quad &\leq C \left(\int_\Omega G_\mu(x, z) d\tau(z) \right)^s.
\end{aligned}$$

Here (3.12) and (3.13) follow from (3.1), (3.14) and (3.15) follow from Hölder inequality, and (3.16) follows from Lemma 2.2, Hölder inequality and (3.10).

Note that the above approach could be applied to the case $p = 1$.

If $0 \leq p < 1$ then

$$\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p] \leq C(\mathbb{G}_\mu[\delta^\gamma] + \mathbb{G}_\mu[\delta^\gamma \mathbb{G}_\mu[\tau]]) \leq C(\mathbb{G}_\mu[\delta^\gamma] + (\mathbb{G}_\mu[\tau])^s)$$

Then (3.11) follows by the a similar argument as in the proof of Lemma 3.3. \square

3.3. Boundary value problem.

Lemma 3.8. *Let $\gamma > -1 - \alpha_+$, $0 < p < p_{\mu, \gamma}^*$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Then there exists a constant $C_1 > 0$ such that (1.15) holds.*

Proof. We first treat the case $p \geq 1$.

Assume that $\sigma = \delta_\xi$, the Dirac measure concentrated at $\xi \in \partial\Omega$. Then by (2.3) and (2.4) we have

$$\begin{aligned}
\mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\delta_\xi])^p](x) &= \int_\Omega \delta(y)^\gamma G_\mu(x, y) (K_\mu(y, \xi))^p dy \\
&\leq \int_\Omega \delta(y)^{\gamma+p\alpha_+} F_\mu(x, y) |y - \xi|^{-p(N+2\alpha_+-2)} dy.
\end{aligned}$$

where

$$F_\mu(x, y) := |x - y|^{2-N} \min \left\{ 1, \delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x - y|^{-2\alpha_+} \right\}, \quad \forall x, y \in \Omega, x \neq y.$$

Without loss of generality we assume that $\xi = 0$. Also we assume that $-1 < \gamma + p\alpha_+ < 0$.

Let $\beta \in (0, \beta^*)$ such that $\delta \in C^2(\overline{\Omega_\beta})$ where Ω_β is given in (2.19). Let $\phi \in C^\infty(\overline{\Omega_{\frac{\beta}{2}}})$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ in $\Omega_{\frac{\beta}{4}}$ and $\phi = 0$ in $\Omega \setminus \overline{\Omega_{\frac{\beta}{2}}}$. Then

$$\begin{aligned} \int_{\Omega} \delta(y)^{\gamma+p\alpha_+} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} dy &= \int_{\Omega} \delta(y)^{\gamma+p\alpha_+} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} \phi(y) dy \\ &\quad + \int_{\Omega} \delta(y)^{\gamma+p\alpha_+} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} (1 - \phi(y)) dy. \end{aligned}$$

By definition of ϕ and using the inequality

$$F_\mu(x, y) \leq \delta(x)^{\alpha_+} |x - y|^{2-\alpha_+-N},$$

we obtain

$$\begin{aligned} \int_{\Omega} \delta(y)^{\gamma+p\alpha_+} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} (1 - \phi(y)) dy \\ = \int_{\Omega \setminus \Omega_{\frac{\beta}{4}}} \delta(y)^{\gamma+p\alpha_+} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} (1 - \phi(y)) dy \\ \leq C(\beta, N, \mu, \gamma) \delta(x)^{\alpha_+}. \end{aligned}$$

Let $\tilde{\beta} \in (0, \frac{\beta}{4})$ be such that $|x - y| > r_0 > 0$ for any $y \in \Omega_{\tilde{\beta}}$. Let $\varepsilon > 0$ be such that

$$p(N + \alpha_+ - 2) = N + \alpha_+ + \gamma - \varepsilon$$

and $0 < \tilde{\varepsilon} < \varepsilon$ be such that

$$\gamma + p\alpha_+ + 1 - \tilde{\varepsilon} > 0.$$

Then, by using the notation in (2.19), we have

$$\begin{aligned} \int_{\Sigma_{\tilde{\beta}}} \delta(y)^{\gamma+p\alpha_++1} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} dS(y) \\ \leq \delta(x)^{\alpha_+} r_0^{2-N-2\alpha_+} \int_{\Sigma_{\tilde{\beta}}} \delta(y)^{\tilde{\varepsilon}} |y|^{-p(N+\alpha_+-2)+\gamma+p\alpha_++1-\tilde{\varepsilon}} dS(y) \\ = \delta(x)^{\alpha_+} r_0^{2-N-2\alpha_+} \int_{\Sigma_{\tilde{\beta}}} \delta(y)^{\tilde{\varepsilon}} |y|^{-N+1+\alpha_+(p-1)+(\varepsilon-\tilde{\varepsilon})} dS(y). \end{aligned}$$

Now note that by the choice of $\tilde{\varepsilon}$ we have $N - 2 - N + 1 + \alpha_+(p - 1) + (\varepsilon - \tilde{\varepsilon}) > -1$, which implies that

$$\sup_{\tilde{\beta} \in (0, \frac{\beta}{4})} \int_{\Sigma_{\tilde{\beta}}} |y|^{-N+1+\alpha_+(p-1)+(\varepsilon-\tilde{\varepsilon})} dS(y) < C.$$

Combining above estimates leads to

$$(3.18) \quad \lim_{\tilde{\beta} \rightarrow 0} \int_{\Sigma_{\tilde{\beta}}} \delta(y)^{\gamma+p\alpha_++1} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} dS(y) = 0.$$

Now note that

$$\begin{aligned}
& - \int_{\Omega_\beta} \nabla \delta(y) \nabla F_\mu(x, y) \delta(y)^{\gamma+p\alpha_++1} |y|^{-p(N+2\alpha_+-2)} \phi(y) dy \\
& = (N-2) \int_{\Omega_\beta} \frac{\nabla \delta(y) \cdot (x-y)}{|x-y|^N} \min \left\{ 1, \delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x-y|^{-2\alpha_+} \right\} |y|^{-p(N+2\alpha_+-2)} \phi(y) dy \\
& - \int_{\Omega_\beta} \nabla \delta(y) \nabla \left(\min \left\{ 1, \delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x-y|^{-2\alpha_+} \right\} \right) |x-y|^{2-N} |y|^{-p(N+2\alpha_+-2)} \phi(y) dy.
\end{aligned}$$

On the other hand

$$\begin{aligned}
& - \nabla_y \delta(y) \nabla_y \left(\min \left\{ 1, \delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x-y|^{-2\alpha_+} \right\} \right) \\
& \leq 2\alpha_+ |x-y|^{-1} \min \left\{ 1, \delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x-y|^{-2\alpha_+} \right\} \quad a.e \text{ in } \Omega.
\end{aligned}$$

By collecting above estimates, we obtain

$$\begin{aligned}
& - \int_{\Omega_\beta} \nabla \delta(y) \nabla F_\mu(x, y) \delta(y)^{\gamma+p\alpha_++1} |y|^{-p(N+2\alpha_+-2)} \phi(y) dy \\
(3.19) \quad & \leq C \delta(x)^{\alpha_+} \int_{\Omega} |x-y|^{-(N+\alpha_+-1)} |y|^{-p(N+\alpha_+-2)+1+\gamma} dy.
\end{aligned}$$

It follows from integration by parts, (3.18) and (3.19) that

$$\begin{aligned}
& \int_{\Omega_\beta} \delta(y)^{\gamma+p\alpha_+} F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} \phi(y) dy \\
& = \frac{1}{\gamma + p\alpha_+ + 1} \int_{\Omega_\beta} \nabla(\delta(y)^{\gamma+p\alpha_++1}) \nabla \delta(y) F_\mu(x, y) |y|^{-p(N+2\alpha_+-2)} \phi(y) dy \\
& \leq C \delta(x)^{\alpha_+} \int_{\Omega} |x-y|^{-(N+2\alpha_+-2)} |y|^{-p(N+\alpha_+-2)+\alpha_++\gamma} dy \\
& + C \delta(x)^{\alpha_+} \int_{\Omega} |x-y|^{-(N+\alpha_+-1)} |y|^{-p(N+\alpha_+-2)+1+\gamma} dy \\
& =: M(x) + N(x).
\end{aligned}$$

We now estimate

$$\begin{aligned}
\frac{M(x)}{K_\mu(x, 0)} & \leq c_3^{q+2} |x|^{N+2\alpha_+-2} \int_{\Omega} |x-y|^{-(N+2\alpha_+-2)} |y|^{-p(N+\alpha_+-2)+\alpha_++\gamma} dy \\
& \leq c_3^{q+2} |x|^{N+\alpha_++\gamma-p(N+\alpha_+-2)} \int_{\mathbb{R}^N} |e_x - \eta|^{-(N+2\alpha_+-2)} |\eta|^{-p(N+\alpha_+-2)+\alpha_++\gamma} d\eta,
\end{aligned}$$

where $e_x = |x|^{-1}x$. The last integral is finite and independent of x since $\gamma \leq 0$, $0 < \alpha_+ < 1$ and $N \geq 3$. Similarly we have

$$\begin{aligned}
\frac{N(x)}{K_\mu(x, 0)} & \leq c_4^{q+2} |x|^{N+2\alpha_+-2} \int_{\Omega} |x-y|^{-(N+\alpha_+-1)} |y|^{-p(N+\alpha_+-2)+1+\gamma} dy \\
& \leq c_3^{q+2} |x|^{N+\alpha_++\gamma-p(N+\alpha_+-2)} \int_{\mathbb{R}^N} |e_x - \eta|^{-(N+\alpha_+-1)} |\eta|^{-p(N+\alpha_+-2)+1+\gamma} d\eta.
\end{aligned}$$

Combining above estimates implies that there exists a positive constant $C = C(\Omega, N, \mu, \gamma) > 0$ such that

$$(3.20) \quad \mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\delta_\xi]^p)](x) \leq C |x - \xi|^{N+\alpha_++\gamma-p(N+\alpha_+-2)} K_\mu(x, \xi), \quad \forall (x, \xi) \in \Omega \times \partial\Omega.$$

If $\gamma + p\alpha_+ \geq 0$, (3.20) can be obtained by an argument similar to the proof of [8, Theorem 3.1].

Finally observe that

$$\mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\nu])^p](x) \leq C \int_{\partial\Omega} \int_{\Omega} \delta(y)^\gamma G_\mu(x, y) (K_\mu(y, \xi))^p dy d\nu(\xi),$$

which, joint with (3.20), yields (1.15).

Next, if $p \in [0, 1)$ then

$$(3.21) \quad \mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\nu])^p] \leq C(\mathbb{G}_\mu[\delta^\gamma] + \mathbb{G}_\mu[\delta^\gamma \mathbb{K}_\mu[\nu]]) \leq C(\mathbb{G}_\mu[\delta^\gamma] + \mathbb{K}_\mu[\nu]).$$

This, combined with the inequality

$$\mathbb{G}_\mu[\delta^\gamma] \leq c\delta^{\alpha_+} \leq c'\mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega,$$

leads to (1.15). \square

Estimate (1.15) is a sufficient condition for the existence of weak solutions of

$$(3.22) \quad \begin{cases} -L_\mu u = \delta^\gamma u^p & \text{in } \Omega, \\ u = \varrho\nu & \text{on } \partial\Omega. \end{cases}$$

Proposition 3.9. *Let $p > 0$, $\gamma > -1 - \alpha_+$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$.*

(i) *Assume that there exists a constant $C_1 > 0$ such that (1.15) holds. Then problem (3.22) admits a solution u satisfying*

$$(3.23) \quad \mathbb{K}_\mu[\varrho\nu] \leq u \leq C'\mathbb{K}_\mu[\varrho\nu] \quad \text{a.e. in } \Omega,$$

with a constant $C' > 0$, for any $\varrho \geq 0$ small enough if $p > 1$, for any $\varrho > 0$ if $p < 1$.

(ii) *Assume that u is a solution of (3.22) and $0 < p < p_{\mu, \gamma}^*$. Then there exists a constant $C'' > 0$ such that*

$$(3.24) \quad \mathbb{K}_\mu[\varrho\nu] \leq u \leq C''(\mathbb{K}_\mu[\varrho\nu] + \delta^{\alpha_+}) \quad \text{a.e. in } \Omega.$$

Proof. The existence and the a priori estimates can be obtained by using an argument as in the proof of Proposition 3.4 and Proposition 3.6. \square

Definition 3.10. *Let $p > 0$ and $\gamma > -1 - \alpha_+$.*

(i) *We say that $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$ is p admissible if and only if the problem (3.7) admits a solution for $\sigma > 0$ small.*

(ii) *We say that $\nu \in \mathfrak{M}^+(\partial\Omega)$ is p admissible if and only if the problem (3.22) admits a solution for $\varrho > 0$ small.*

The above results allow to study elliptic equations with interior and boundary measures.

Proposition 3.11. *Let $p > 0$, $\gamma \in (\frac{-2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$, $\sigma > 0$, $\varrho > 0$, $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ are p admissible. Then for $\sigma > 0$ and $\varrho > 0$ small enough problem (1.16) has a solution u satisfying*

$$(3.25) \quad u \leq C(\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu]) \quad \text{a.e. in } \Omega$$

with a constant $C > 0$.

Furthermore when $0 < p < p_{\mu, \gamma}^$ there exists another constant $C > 0$ such that if u is a solution of (1.16) then the following a priori estimate holds*

$$(3.26) \quad \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] \leq u \leq C(\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] + \delta^{\alpha_+}) \quad \text{a.e. in } \Omega.$$

Proof. The argument used in the proof of [8, Theorem 3.13] can be applied. Put $v := u - \mathbb{K}_\mu[\varrho\nu]$ then v satisfies

$$(3.27) \quad \begin{cases} -L_\mu v = \delta^\gamma(v + \mathbb{K}_\mu[\varrho\nu])^p + \sigma\tau & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the following problem

$$(3.28) \quad \begin{cases} -L_\mu w = c_p \delta^\gamma w^p + c_p \delta^\gamma (\mathbb{K}_\mu[\varrho\nu])^p + \sigma\tau & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

where $c_p := \max\{1, 2^{p-1}\}$. Since ν is p admissible, it follows that $\delta^\gamma(\mathbb{K}_\mu[\varrho\nu])^p \in L^1(\Omega; \delta^{\alpha+})$. Since τ is p admissible, we deduce that problem (3.28) admits a solution w for $\sigma > 0$ and $\varrho > 0$ small enough. Notice that w is a supersolution of (3.28), we infer that there is a solution v of (3.27) satisfying $v \leq w$ a. e. in Ω . Since ν is p admissible, there exists a constant c such that

$$(3.29) \quad \mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\varrho\nu])^p] \leq c\mathbb{K}_\mu[\varrho\nu] \quad \text{a.e. in } \Omega.$$

By Proposition 3.4, we deduce that

$$w \leq c' \mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\varrho\nu])^p + \sigma\tau] \quad \text{a.e. in } \Omega.$$

This, together with (3.29), leads to (3.25).

If $0 < p < p_{\mu,\gamma}^*$ then (3.26) follows from Proposition 3.6 and Proposition 3.9. \square

Proof of Theorem A. Estimate (1.14) and (1.15) are proved in Lemma 3.8 and Lemma 3.3 respectively. The rest of the proof follows from Proposition 3.11. \square

3.4. Capacities and existence results. For $a > -1$, $0 \leq \alpha \leq \beta < N$ and $s > 1$, let $N_{\alpha,\beta}$, $\mathbb{N}_{\alpha,\beta}$ and $\text{Cap}_{\mathbb{N}_{\alpha,\beta},s}^a$ be defined as in (1.19), (1.20) and (1.21) respectively.

In this section, we are interested in the solvability of the following integral equations

$$u = \mathbb{N}_{\alpha,\beta}[\delta^a u^p] + \mathbb{N}_{\alpha,\beta}[\tau],$$

where $p > 1$ and $\tau \in \mathfrak{M}^+(\overline{\Omega})$. To this end, we will adapt the argument in [9, Section 2].

Let \mathbf{Z} be a metric space and $\omega \in \mathfrak{M}^+(\mathbf{Z})$. Let $J : \mathbf{Z} \times \mathbf{Z} \rightarrow (0, \infty]$ be a Borel positive kernel such that J is symmetric and J^{-1} satisfies a quasi-metric inequality, i.e. there is a constant $C > 1$ such that for all $x, y, z \in \mathbf{Z}$,

$$\frac{1}{J(x, y)} \leq C \left(\frac{1}{J(x, z)} + \frac{1}{J(z, y)} \right).$$

Under these conditions, one can define the quasi-metric d by

$$d(x, y) := \frac{1}{J(x, y)}$$

and denote by $B_r(x) := \{y \in \mathbf{Z} : d(x, y) < r\}$ the open d -ball of radius $r > 0$ and center x . Note that this set can be empty.

For $\omega \in \mathfrak{M}^+(\mathbf{Z})$ we define the potentials $\mathbb{J}[\omega]$ and $\mathbb{J}[\phi, \omega]$ by

$$\mathbb{J}[\omega](x) := \int_{\mathbf{Z}} J(x, y) d\omega(y) \quad \text{and} \quad \mathbb{J}[\phi, \omega](x) := \int_{\mathbf{Z}} J(x, y) \phi(y) d\omega(y).$$

For $t > 1$ the capacity $\text{Cap}_{\mathbb{J},t}^\omega$ in \mathbf{Z} is defined by

$$\text{Cap}_{\mathbb{J},t}^\omega(E) := \inf \left\{ \int_{\mathbf{Z}} \phi(x)^t d\omega(x) : \phi \geq 0, \mathbb{J}[\phi, \omega] \geq \chi_E \right\},$$

for any Borel $E \subset \mathbf{Z}$.

Proposition 3.12. ([20]) *Let $p > 1$ and $\tau, \omega \in \mathfrak{M}^+(\mathbf{Z})$ such that*

$$(3.30) \quad \int_0^{2r} \frac{\omega(B_s(x))}{s^2} ds \leq C \int_0^r \frac{\omega(B_s(x))}{s^2} ds,$$

$$(3.31) \quad \sup_{y \in B_r(x)} \int_0^r \frac{\omega(B_s(y))}{s^2} ds \leq C \int_0^r \frac{\omega(B_s(x))}{s^2} ds,$$

for any $r > 0, x \in \mathbf{Z}$, where $C > 0$ is a constant. Then the following statements are equivalent:

1. The equation $u = \mathbb{J}[u^q, \omega] + \sigma \mathbb{J}[\tau]$ has a solution for $\sigma > 0$ small.
2. The inequality

$$\int_E \mathbb{J}[\tau_E]^p d\omega \leq C\tau(E),$$

holds for any Borel set $E \subset \mathbf{Z}$ where $\tau_E = \chi_E \tau$.

3. For any Borel set $E \subset \mathbf{Z}$ there holds

$$\tau(E) \leq C \text{Cap}_{\mathbb{J}, p'}^\omega(E).$$

4. The inequality

$$\mathbb{J}[(\mathbb{J}[\tau])^p, \omega] \leq C\mathbb{J}[\tau] < \infty \quad \omega - a.e. \text{ holds.}$$

We will point out below that $N_{\alpha, \beta}$ satisfies all assumptions of \mathbb{J} in Proposition 3.12.

Proposition 3.13. ([9, Lemma 2.2]) *$N_{\alpha, \beta}$ is symmetric and satisfies the quasi-metric inequality.*

Next we give sufficient conditions for (3.30), (3.31) to hold.

Proposition 3.14. *Let $\omega = \delta(x)^a \chi_\Omega(x) dx$ with $a > -1$. Then (3.30) and (3.31) hold.*

Proof. If $a \geq 0$ the the statement follows directly from [9, Lemma 2.3]. We are left with the case $-1 < a < 0$.

We claim that for any $0 < s < 8\text{diam}(\overline{\Omega})$ and any $x \in \overline{\Omega}$ we have

$$(3.32) \quad \nu(B_s(x)) \sim \max\{\delta(x), s\}^a s^N.$$

Indeed, in order to obtain (3.32), we consider four cases.

Case 1: $4s \leq \delta(x)$. Then $\delta(x) \sim \delta(y)$ for any $y \in B_s(x)$ and the proof of (3.32) can be obtained easily.

Case 2: $s > \frac{\delta(x)}{4}$. Then $\delta(y) \leq 5s$ thus

$$\int_{B_s(x) \cap \overline{\Omega}} \delta(y)^a dy \geq C s^{a+N} \geq C \max\{\delta(x), s\}^a s^N.$$

Case 3: $\frac{\delta(x)}{4} \leq s \leq 4\delta(x)$. Since Ω is smooth there exists a $r^* > 0$ such that

$$(3.33) \quad \int_{B_{r_0}(x_i) \cap \overline{\Omega}} \delta(y)^a dy \leq \frac{C}{a+1} r_0^{a+N}, \quad \forall r_0 \leq \frac{r^*}{8} \text{ and } \delta(x_i) < \frac{r^*}{4}.$$

Set $r_0 := r^* \frac{\delta(x)}{32\text{diam}(\overline{\Omega})}$, then there exists $x_i \in B_s(x)$ and $i = 1, \dots, k$ such that

$$B_s(x) \subset \cup_{i=1}^k B_{r_0}(x_i).$$

We note that k does not depend neither on x , nor on $\delta(x)$. Thus we have

$$\int_{B_s(x) \cap \overline{\Omega}} \delta(y)^a dy \leq \sum_{i=1}^k \int_{B_{r_0}(x_i) \cap \overline{\Omega}} \delta(y)^a dy.$$

Now by (3.33) we get

$$\int_{B_{r_0}(x_i) \cap \overline{\Omega}} \delta(y)^a dy \leq C \delta(x)^{a+N} \leq C \max\{\delta(x), s\}^a s^N, \quad \text{if } \delta(x_i) < \frac{r^*}{4}$$

and

$$\int_{B_{r_0}(x_i) \cap \overline{\Omega}} \delta(y)^a dy \leq C (r^*)^a \delta(x)^N \leq C \max\{\delta(x), s\}^a s^N, \quad \text{if } \delta(x_i) \geq \frac{r^*}{4}$$

and hence (3.32) follows.

Case 4: $s \geq 4\delta(x)$. Set $r_0 := r^* \frac{s}{32 \text{diam}(\overline{\Omega})}$, then the proof of (3.32) follows due to a similar argument as in Case 3.

The rest of the proof can be proceeded as in the proof of [9, Lemma 2.3] and we omit it. \square

We recall below the definition of the capacity associated to $\mathbb{N}_{\alpha,\beta}$ (see [20]).

Definition 3.15. Let $a > -1$, $0 \leq \alpha \leq \beta < N$ and $s > 1$. Define $\text{Cap}_{\mathbb{N}_{\alpha,\beta},s}^a$ by

$$\text{Cap}_{\mathbb{N}_{\alpha,\beta},s}^a(E) := \inf \left\{ \int_{\overline{\Omega}} \delta^a \phi^s dy : \phi \geq 0, \mathbb{N}_{\alpha,\beta}[\delta^a \phi] \geq \chi_E \right\},$$

for any Borel set $E \subset \overline{\Omega}$.

Clearly we have

$$\text{Cap}_{\mathbb{N}_{\alpha,\beta},s}^a(E) = \inf \left\{ \int_{\overline{\Omega}} \delta^{-a(s-1)} \phi^s dy : \phi \geq 0, \mathbb{N}_{\alpha,\beta}[\phi] \geq \chi_E \right\},$$

for any Borel set $E \subset \overline{\Omega}$. Furthermore we have by [1, Theorem 2.5.1]

$$\left(\text{Cap}_{\mathbb{N}_{\alpha,\beta},s}^a(E) \right)^{\frac{1}{s}} = \inf \left\{ \omega(E) : \omega \in \mathfrak{M}_b^+(\overline{\Omega}), \|\mathbb{N}_{\alpha,\beta}[\omega]\|_{L^{s'}(\overline{\Omega}; \delta^a)} \leq 1 \right\},$$

for any compact set $E \subset \overline{\Omega}$ where s' is the conjugate exponent of s .

Thanks to Proposition 3.13 and Proposition 3.14, we can apply Proposition 3.12 to obtain

Proposition 3.16. Let $\tau \in \mathfrak{M}^+(\overline{\Omega})$, $a > -1$, $0 \leq \alpha \leq \beta < N$ and $p > 1$. Then the following statements are equivalent:

1. The equation $u = \mathbb{N}_{\alpha,\beta}[\delta^a u^p] + \sigma \mathbb{N}_{\alpha,\beta}[\tau]$ has a solution for some $\sigma > 0$.
2. The inequality

$$\int_E \delta^a (\mathbb{N}_{\alpha,\beta}[\tau_E])^p dx \leq C \tau(E),$$

holds for any Borel set $E \subset \overline{\Omega}$ where $\tau_E = \chi_E \tau$.

3. For any compact set $K \subset \overline{\Omega}$, there holds

$$\tau(K) \leq C \text{Cap}_{\mathbb{N}_{\alpha,\beta},p'}^a(K).$$

4. The inequality

$$\mathbb{N}_{\alpha,\beta}[\delta^a (\mathbb{N}_{\alpha,\beta}[\tau])^p] \leq C \mathbb{N}_{\alpha,\beta}[\tau] < \infty \quad \text{a.e. in } \overline{\Omega},$$

holds for some $C > 0$.

We next recall the capacity $\text{Cap}_{\theta,s}^{\partial\Omega}$ introduced in [9] which can be employed to deal with boundary measures. Let $\theta \in (0, N-1)$ and denote by \mathcal{B}_θ the Bessel kernel in \mathbb{R}^{N-1} with order θ . For $s > 1$, define

$$(3.34) \quad \text{Cap}_{\mathcal{B}_\theta,s}(F) := \inf \left\{ \int_{\mathbb{R}^{N-1}} \delta^s dy : \phi \geq 0, \mathcal{B}_\theta * \phi \geq \chi_F \right\}$$

for any Borel set $F \subset \mathbb{R}^{N-1}$. Since Ω be a bounded smooth domain in \mathbb{R}^N , there exist open sets O_1, \dots, O_m in \mathbb{R}^N , diffeomorphisms $T_i : O_i \rightarrow B_1(0)$ and compact sets K_1, \dots, K_m in $\partial\Omega$ such that

- (i) $K_i \subset O_i$, $1 \leq i \leq m$ and $\partial\Omega \cup_{i=1}^m K_i$,
- (ii) $T_i(O_i \cap \partial\Omega) = B_1(0) \cap \{x_N = 0\}$, $T_i(O_i \cap \Omega) = B_1(0) \cap \{x_N > 0\}$,
- (iii) For any $x \in O_i \cap \Omega$, there exists $y \in O_i \cap \partial\Omega$ such that $\delta(x) = |x - y|$. We then define the $\text{Cap}_{\theta,s}^{\partial\Omega}$ -capacity of a compact set $F \subset \partial\Omega$ by

$$(3.35) \quad \text{Cap}_{\theta,s}^{\partial\Omega}(F) := \sum_{i=1}^m \text{Cap}_{B_\theta,s}(\tilde{T}_i(F \cap K_i)),$$

where $T_i(F \cap K_i) = \tilde{T}_i(F \cap K_i) \times \{x_N = 0\}$.

The following result is obtained by the same argument as in the proof of [9, Proposition 2.9].

Proposition 3.17. *Let $a > -1$, $0 \leq \alpha \leq \beta < N$ and $s > 1$. Assume that $-1 + s'(1 + \alpha - \beta) < a < -1 + s'(N + \alpha - \beta)$. Then there holds*

$$\text{Cap}_{N_{\alpha,\beta},s}^a(E) \sim \text{Cap}_{\beta-\alpha+\frac{a+1}{s'}-1,s}^{\partial\Omega}(E),$$

for any Borel $E \subset \partial\Omega$.

By using Proposition 3.16, the fact that $(p+1)\alpha_+ + \gamma > -1$ for $p > 1$ and $\gamma > -2$ and by adapting the argument in the proof of [9, Theorem 1.2], we obtain easily the following results whose proofs are omitted.

Proposition 3.18. *Let $p > 1$, $\gamma > -2$, $\sigma > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$. Then the following statements are equivalent:*

1. *There exists $C > 0$ such that the following inequality hold*

$$\tau(E) \leq C \text{Cap}_{N_{2\alpha_+,2},p'}^{(p+1)\alpha_++\gamma}(E),$$

for any Borel $E \subset \overline{\Omega}$.

2. *There exists a constant $C_2 > 0$ such that (1.14) holds.*
3. *Problem (3.7) has a positive weak solution for $\sigma > 0$ small enough.*

Combining Proposition 3.16 and Proposition 3.17 leads to the following result.

Proposition 3.19. *Let $p > 1$, $-2 < \gamma < -1 - \alpha_+ + p(N + \alpha_+ - 2)$, $\varrho > 0$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Then, the following statements are equivalent:*

1. *There exists $C > 0$ such that the following inequality hold*

$$\nu(F) \leq C \text{Cap}_{1-\alpha_++\frac{1+\alpha_++\gamma}{p},p'}^{\partial\Omega}(F),$$

for any Borel $F \subset \partial\Omega$.

2. *There exists $C_1 > 0$ such that (1.15) holds.*
3. *Problem (3.22) has a positive weak solution for $\varrho > 0$ small enough.*

Proof of Theorem B. The implications (i) \iff (ii) \implies (iii) follow from Proposition 3.19, Proposition 3.18 and Proposition 3.11. We will show that (iii) \implies (ii). Since (1.16) has a weak solution for $\sigma > 0$ small and $\varrho > 0$ small, it follows that (3.7) admits a solution for $\sigma > 0$ small and (3.22) admits a solution for $\varrho > 0$ small. Due to Proposition 3.19 and Proposition 3.18, we derive (1.14) and (1.15). This completes the proof. \square

4. ELLIPTIC SYSTEMS: THE POWER CASE

Let $\mu \in (0, \frac{1}{4}]$. In this section, we deal with system (1.27). We recall that

$$q := \tilde{p} \frac{p+1}{\tilde{p}+1}, \quad \tilde{q} := p \frac{\tilde{p}+1}{p+1}.$$

Without loss of generality, we can assume that $0 < p \leq \tilde{p}$. Then $p \leq q \leq \tilde{q} \leq \tilde{p}$ if $p\tilde{p} \geq 1$.

Put

$$t_\gamma := \tilde{p} \left(p - \frac{2+\gamma}{N+\alpha_+-2} \right).$$

If $q < q_{\mu,\gamma}^*$ then $t_\gamma < q < p_{\mu,\gamma}^*$.

Lemma 4.1. *Let $p > 0$, $\tilde{p} > 0$, $\{\gamma, \tilde{\gamma}\} \subset (-\frac{2(N+\alpha_+-1)\alpha_+}{N+2\alpha_+-2}, \frac{2\alpha_+}{N-2})$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$. Assume $q < \min(p_{\mu,\gamma}^*, p_{\mu,\tilde{\gamma}}^*)$. Then for any $t \in (\max(0, t_\gamma), \tilde{p}]$, there exists a positive constant $c = c(N, p, \tilde{p}, \mu, \gamma, \tilde{\gamma}, t, \tau)$ (independent of τ if $p > 1$) such that*

$$(4.1) \quad (\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p])^{\tilde{p}} \leq c(\mathbb{G}_\mu[\tau])^t.$$

In particular,

$$(4.2) \quad (\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p])^{\tilde{p}} \leq C(\mathbb{G}_\mu[\tau])^q,$$

$$(4.3) \quad \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p])^{\tilde{p}}] \leq C\mathbb{G}_\mu[\tau]$$

where $C = C(N, p, \tilde{p}, \mu, \gamma, \tilde{\gamma}, \tau)$.

Proof. Since $q < p_{\mu,\gamma}^*$, it follows that $p < p_{\mu,\gamma}^*$, hence $\max(0, p - \frac{2+\gamma}{N-2+\alpha_+}) < 1$. Let $t \in (\max(0, t_\gamma), \tilde{p}]$ then $\max(0, p - \frac{2+\gamma}{N-2+\alpha_+}) < \frac{t}{\tilde{p}} \leq 1$. By applying Lemma 3.7 with s replaced by $\frac{t}{\tilde{p}}$ respectively in order to obtain

$$\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p] \leq c(\mathbb{G}_\mu[\tau])^{\frac{t}{\tilde{p}}},$$

which implies (4.1). Since $t_\gamma < q \leq \tilde{p}$, by taking $t = q$ in (4.1) we obtain (4.2). Next, since $q < p_{\mu,\tilde{\gamma}}^*$, by apply Lemma 3.7 with γ replaced by $\tilde{\gamma}$ and (4.2), we get

$$\mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\tau])^p])^{\tilde{p}}] \leq C\mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\tau])^q] \leq C\mathbb{G}_\mu[\tau].$$

□

Lemma 4.2. *Let $p > 0$, $\tilde{p} > 0$, $\gamma > -2$, $\tilde{\gamma} > -2$, $\tau, \tilde{\tau} \in \mathfrak{M}^+(\Omega, \delta^{\alpha_+})$ and $\nu, \tilde{\nu} \in \mathfrak{M}^+(\partial\Omega)$. Assume that there exist positive functions $U \in L^{\tilde{p}}(\Omega; \delta^{\alpha_++\tilde{\gamma}})$ and $V \in L^p(\Omega; \delta^{\alpha_++\gamma})$ such that*

$$(4.4) \quad \begin{aligned} U &\geq \mathbb{G}_\mu[\delta^\gamma(V + \mathbb{K}_\mu[\tilde{\nu}])^p] + \mathbb{G}_\mu[\sigma\tau], \\ V &\geq \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(U + \mathbb{K}_\mu[\nu])^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] \end{aligned}$$

in Ω . Then there exists a weak solution (u, v) of (1.27) such that

$$(4.5) \quad \begin{aligned} \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\nu] &\leq u \leq U, \\ \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\nu}] &\leq v \leq V. \end{aligned}$$

Proof. Put $u_0 := 0$ and

$$(4.6) \quad \begin{cases} v_{n+1} := \mathbb{G}_\mu[\delta^{\tilde{\gamma}}u_n^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\nu}], & n \geq 0, \\ u_n := \mathbb{G}_\mu[\delta^\gamma v_n^p] + \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\nu], & n \geq 1. \end{cases}$$

We see that $0 \leq v_1 = \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\nu}] \leq V$. It is easy to see that $\{u_n\}$ and $\{v_n\}$ are nondecreasing sequences, $0 \leq u_n \leq U$ and $0 \leq v_n \leq V$ in Ω . By monotone convergence theorem, there exist

$u \in L^{\tilde{p}}(\Omega; \delta^{\alpha_+ + \tilde{\gamma}})$ and $v \in L^p(\Omega; \delta^{\alpha_+ + \gamma})$ such that $u_n \rightarrow u$ in $L^1(\Omega)$, $v_n \rightarrow v$ in $L^1(\Omega)$, $u_n^{\tilde{p}} \rightarrow u^{\tilde{p}}$ in $L^1(\Omega; \delta^{\alpha_+ + \tilde{\gamma}})$, $v_n^p \rightarrow v^p$ in $L^1(\Omega; \delta^{\alpha_+ + \gamma})$. Moreover $u \leq U$ and $v \leq V$ in Ω . By letting $n \rightarrow \infty$ in (4.6), we obtain

$$(4.7) \quad \begin{cases} v = \mathbb{G}_\mu[\delta^{\tilde{\gamma}} u^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma} \tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}], \\ u = \mathbb{G}_\mu[\delta^\gamma v^p] + \mathbb{G}_\mu[\sigma \tau] + \mathbb{K}_\mu[\varrho \nu]. \end{cases}$$

Thus (u, v) is a weak solution of (1.27) and satisfies (4.5). \square

Proof of Theorem C. We first show that the follow system has weak a solution

$$(4.8) \quad \begin{cases} -L_\mu w = \delta^\gamma(\tilde{w} + \mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}])^p + \sigma \tau & \text{in } \Omega, \\ -L_\mu \tilde{w} = \delta^{\tilde{\gamma}}(\tilde{w} + \mathbb{K}_\mu[\varrho \nu])^{\tilde{p}} + \tilde{\sigma} \tilde{\tau} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix $\vartheta_i > 0$, ($i = 1, 2, 3, 4$) and set

$$\Psi := \delta^{\tilde{\gamma}} \left\{ \mathbb{G}_\mu[\vartheta_1 \tau + \delta^\gamma(\mathbb{K}_\mu[\vartheta_2 \tilde{\nu}])^p] \right\}^{\tilde{p}} + \delta^{\tilde{\gamma}}(\mathbb{K}_\mu[\vartheta_3 \nu])^{\tilde{p}} + \vartheta_4 \tilde{\tau}.$$

For $\kappa \in (0, 1]$, put

$$\sigma := \kappa^{\frac{1}{\tilde{p}}} \vartheta_1, \quad \tilde{\sigma} := \kappa \vartheta_4, \quad \varrho := \kappa^{\frac{1}{\tilde{p}}} \vartheta_3, \quad \tilde{\varrho} := \kappa^{\frac{1}{p\tilde{p}}} \vartheta_2.$$

Then from the assumption, we deduce that $\Psi \in \mathfrak{M}^+(\Omega, \delta^{\alpha_+})$. By Lemma 4.1,

$$(4.9) \quad \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\Psi])^p])^{\tilde{p}}] \leq C \mathbb{G}_\mu[\Psi]$$

where $C = C(N, p, \tilde{p}, \mu, \sigma, \tilde{\sigma}, \kappa, \tau, \tilde{\tau})$. Set

$$V := a_3 \mathbb{G}_\mu[\kappa \Psi] \quad \text{and} \quad U := \mathbb{G}_\mu[\delta^\gamma(V + \mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}])^p + \sigma \tau]$$

where a_3 will be determined later on. We have

$$\begin{aligned} & \delta^{\tilde{\gamma}}(U + \mathbb{K}_\mu[\varrho \nu])^{\tilde{p}} + \tilde{\sigma} \tilde{\tau} \\ & \leq c(\tilde{p}) \delta^{\tilde{\gamma}} \left\{ \left(\mathbb{G}_\mu \left[\delta^\gamma(a_3 \kappa \mathbb{G}_\mu[\Psi] + \mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}])^p \right] \right)^{\tilde{p}} + (\mathbb{G}_\mu[\sigma \tau])^{\tilde{p}} + (\mathbb{K}_\mu[\varrho \nu])^{\tilde{p}} \right\} + \tilde{\sigma} \tilde{\tau} \\ & \leq c(p, \tilde{p}) \delta^{\tilde{\gamma}} \left\{ a_3^{p\tilde{p}} \kappa^{p\tilde{p}} \left(\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\Psi])^p] \right)^{\tilde{p}} + \left(\mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}])^p] \right)^{\tilde{p}} \right\} \\ & \quad + c(\tilde{p}) \delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\sigma \tau])^{\tilde{p}} + c(\tilde{p}) \delta^{\tilde{\gamma}}(\mathbb{K}_\mu[\varrho \nu])^{\tilde{p}} + \tilde{\sigma} \tilde{\tau}. \end{aligned}$$

It follows that

$$(4.10) \quad \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(U + \mathbb{K}_\mu[\varrho \nu])^{\tilde{p}} + \tilde{\sigma} \tilde{\tau}] \leq I_1 + I_2$$

where

$$I_1 := c(p, \tilde{p}) \left\{ a_3^{p\tilde{p}} \kappa^{p\tilde{p}} \mathbb{G}_\mu \left[\delta^{\tilde{\gamma}} \left(\mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\Psi])^p] \right)^{\tilde{p}} \right] + \mathbb{G}_\mu \left[\delta^{\tilde{\gamma}} \left(\mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}])^p] \right)^{\tilde{p}} \right] \right\},$$

$$I_2 := c(\tilde{p}) \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\sigma \tau])^{\tilde{p}}] + c(\tilde{p}) \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{K}_\mu[\varrho \nu])^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma} \tilde{\tau}].$$

We first estimate I_1 . Observe that

$$\begin{aligned} \mathbb{G}_\mu \left[\delta^{\tilde{\gamma}} \left(\mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}])^p] \right)^{\tilde{p}} \right] &= \mathbb{G}_\mu \left[\kappa \delta^{\tilde{\gamma}} \left(\mathbb{G}_\mu[\delta^\gamma(\mathbb{K}_\mu[\vartheta_2 \tilde{\nu}])^p] \right)^{\tilde{p}} \right] \\ &\leq \mathbb{G}_\mu[\kappa \Psi]. \end{aligned}$$

This, together with (4.9) implies

$$(4.11) \quad I_1 \leq c(a_3^{p\tilde{p}} \kappa^{p\tilde{p}-1} + 1) \mathbb{G}_\mu[\kappa \Psi].$$

Next it is easy to see that

$$(4.12) \quad I_2 \leq c(\tilde{p})\mathbb{G}_\mu[\kappa\Psi].$$

By collecting (4.10), (4.11) and (4.12), we obtain

$$(4.13) \quad \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(U + \mathbb{K}_\mu[\varrho\nu])^{\tilde{p}} + \tilde{\sigma}\tilde{\tau}] \leq c(a_3^{p\tilde{p}}\kappa^{p\tilde{p}-1} + 1)\mathbb{G}_\mu[\kappa\Psi]$$

with another constant c . We will choose a_3 and κ such that

$$(4.14) \quad c(a_3^{p\tilde{p}}\kappa^{p\tilde{p}-1} + 1) \leq a_3.$$

If $p\tilde{p} > 1$ then we can choose $a_3 > 0$ large enough and then choose $\kappa > 0$ small enough (depending on a_3) such that (4.14) holds. If $p\tilde{p} < 1$ then for any $\kappa > 0$ there exists a_3 large enough such that (4.14) holds. For such a_3 and κ , we obtain

$$\mathbb{G}_\mu[\delta^{\tilde{\gamma}}(U + \mathbb{K}_\mu[\varrho\nu])^{\tilde{p}} + \tilde{\sigma}\tilde{\tau}] \leq V.$$

By Lemma 4.2, there exists a weak solution (w, \tilde{w}) of (4.8) for $\sigma > 0$, $\tilde{\sigma} > 0$, $\nu > 0$, $\tilde{\nu} > 0$ small if $p\tilde{p} > 1$, for any $\sigma > 0$, $\tilde{\sigma} > 0$, $\nu > 0$, $\tilde{\nu} > 0$ if $p\tilde{p} < 1$. Moreover, (w, \tilde{w}) satisfies

$$(4.15) \quad \tilde{w} \sim \mathbb{G}_\mu[\omega],$$

$$(4.16) \quad w \sim \mathbb{G}_\mu[\delta^\gamma(\mathbb{G}_\mu[\omega] + \mathbb{K}_\mu[\tilde{\nu}])^p] + \mathbb{G}_\mu[\tau]$$

where $C = C(N, p, \tilde{p}, \mu, \Omega, \sigma, \tilde{\sigma}, \tau, \tilde{\tau})$.

Next put $u := w + \mathbb{K}_\mu[\varrho\nu]$ and $v := \tilde{w} + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}]$ then (u, v) is a weak solution of (1.27). Moreover (1.28) and (1.29) follow directly from (4.15) and (4.16). \square

Proof of Theorem D. Put $\tau^* := \max\{\tau, \tilde{\tau}\}$ and $\nu^* := \max\{\nu, \tilde{\nu}\}$. Fix $\vartheta > 0$, $\tilde{\vartheta} > 0$ and for $\kappa \in (0, 1]$, put $\sigma = \varrho = (\kappa\vartheta)^{\frac{1}{\tilde{p}}}$ and $\tilde{\sigma} = \kappa\tilde{\vartheta}$, $\tilde{\varrho} = (\kappa\tilde{\vartheta})^{\frac{1}{p\tilde{p}}}$. Set

$$\tau^\# := \vartheta\tau + \tilde{\vartheta}\tilde{\tau} \quad \text{and} \quad \nu^\# := \vartheta\nu + \tilde{\vartheta}\tilde{\nu}$$

then $\tau^\# \leq (\vartheta + \tilde{\vartheta})\tau^*$ and $\nu^\# \leq (\vartheta + \tilde{\vartheta})\nu^*$.

Put $V := a_4(\mathbb{G}_\mu[\kappa\tau^\#] + \mathbb{K}_\mu[\kappa\nu^\#])$ where a_4 will be determined later on and put $U := \mathbb{G}_\mu[\delta^\gamma(V + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}])^p + \sigma\tau]$.

We have

$$\begin{aligned} \delta^{\tilde{\gamma}}U^{\tilde{p}} + \tilde{\sigma}\tilde{\tau} &\leq c(p, \tilde{p}) \left\{ a_4^{p\tilde{p}}\kappa^{p\tilde{p}}\delta^{\tilde{\gamma}} \left(\mathbb{G}_\mu[\delta^p(\mathbb{G}_\mu[\tau^\#])^p]^{\tilde{p}} + \mathbb{G}_\mu[\delta^p(\mathbb{K}_\mu[\nu^\#])^p]^{\tilde{p}} \right) \right. \\ &\quad \left. + \sigma^{\tilde{p}}\delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\tau])^{\tilde{p}} + \varrho^{\tilde{p}}\delta^{\tilde{\gamma}}(\mathbb{K}_\mu[\nu])^{\tilde{p}} + \tilde{\sigma}\tilde{\tau} \right\}. \end{aligned}$$

It follows that

$$(4.17) \quad \mathbb{G}_\mu[\delta^{\tilde{\gamma}}(U + \mathbb{K}_\mu[\varrho\nu])^{\tilde{p}} + \tilde{\sigma}\tilde{\tau}] \leq c(p, \tilde{p})(J_1 + J_2)$$

where

$$\begin{aligned} J_1 &:= a_4^{p\tilde{p}}\kappa^{p\tilde{p}} \left(\mathbb{G}_\mu[\delta^{\tilde{\gamma}}\mathbb{G}_\mu[\delta^p(\mathbb{G}_\mu[\tau^\#])^p]^{\tilde{p}}] + \mathbb{G}_\mu[\delta^{\tilde{\gamma}}\mathbb{G}_\mu[\delta^p(\mathbb{K}_\mu[\nu^\#])^p]^{\tilde{p}}] \right), \\ J_2 &:= \sigma^{\tilde{p}}\mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{G}_\mu[\tau])^{\tilde{p}}] + \varrho^{\tilde{p}}\mathbb{G}_\mu[\delta^{\tilde{\gamma}}(\mathbb{K}_\mu[\nu])^{\tilde{p}}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}] + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]. \end{aligned}$$

We first estimate J_1 . We have

$$J_1 \leq a_4^{p\tilde{p}}\kappa^{p\tilde{p}}(\vartheta + \tilde{\vartheta})^{p\tilde{p}} \left(\mathbb{G}_\mu[\delta^{\tilde{\gamma}}\mathbb{G}_\mu[\delta^p(\mathbb{G}_\mu[\tau^*])^p]^{\tilde{p}}] + \mathbb{G}_\mu[\delta^{\tilde{\gamma}}\mathbb{G}_\mu[\delta^p(\mathbb{K}_\mu[\nu^*])^p]^{\tilde{p}}] \right).$$

By (1.30), 1.31 and Proposition 3.19, Proposition 3.18 we infer that

$$J_1 \leq c a_4^{p\tilde{p}}\kappa^{p\tilde{p}}(\vartheta + \tilde{\vartheta})^{p\tilde{p}}(\mathbb{G}_\mu[\tau^*] + \mathbb{K}_\mu[\nu^*])$$

where c is a positive constant. Therefore

$$(4.18) \quad J_1 \leq c a_4^{p\tilde{p}} \kappa^{p\tilde{p}} (\vartheta + \tilde{\vartheta})^{p\tilde{p}} \max(\vartheta^{-1}, \tilde{\vartheta}^{-1}) (\mathbb{G}_\mu[\tau^\#] + \mathbb{K}_\mu[\nu^\#]).$$

We next estimate I_2 . Again by (1.30), 1.31 and Proposition 3.19, Proposition 3.18, we deduce

$$(4.19) \quad \begin{aligned} J_2 &\leq c (\sigma^{\tilde{p}} \mathbb{G}_\mu[\tau] + \varrho^{\tilde{p}} \mathbb{K}_\mu[\nu] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}] + \tilde{\varrho} \mathbb{K}_\mu[\tilde{\nu}]) \\ &= c \kappa (\mathbb{G}_\mu[\tau^\#] + \mathbb{K}_\mu[\nu^\#]). \end{aligned}$$

Combining (4.17), (4.18) and (4.19) implies

$$(4.20) \quad \mathbb{G}_\mu[\delta^{\tilde{\gamma}} U^{\tilde{p}} + \tilde{\sigma} \tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}] \leq C (a_4^{p\tilde{p}} \kappa^{p\tilde{p}-1} + 1) (\mathbb{G}_\mu[\kappa \tau^\#] + \mathbb{K}_\mu[\kappa \nu^\#])$$

where C is another positive constant. We choose $a_4 > 0$ and $\kappa > 0$ such that

$$(4.21) \quad C (a_4^{p\tilde{p}} \kappa^{p\tilde{p}-1} + 1) \leq a_4.$$

Since $p\tilde{p} > 1$, one can choose a_4 large enough and then choose $\kappa > 0$ small enough such that (4.21) holds. For such a_4 and κ , we have

$$\mathbb{G}_\mu[\delta^{\tilde{\gamma}} U^{\tilde{p}} + \tilde{\sigma} \tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho} \tilde{\nu}] \leq V.$$

By Lemma 4.2, there exists a weak solution (u, v) of (1.27) which satisfies (1.32). \square

5. GENERAL NONLINEARITIES

5.1. Absorption case. In this section we treat system (1.33) with $\epsilon = -1$. We recall that $\Lambda_{g,\gamma}$ and $\Lambda_{\tilde{g},\tilde{\gamma}}$ are defined in (1.34).

Proof of Theorem E.

Step 1: We claim that

$$(5.1) \quad \int_{\Omega} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|]) \delta^{\alpha_+ + \gamma} dx + \int_{\Omega} \tilde{g}(\mathbb{K}_\mu[|\nu|] + \mathbb{G}_\mu[|\tau|]) \delta^{\alpha_+ + \tilde{\gamma}} dx < \infty.$$

For $\lambda > 0$, set $\tilde{A}_\lambda := \{x \in \Omega : \mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|] > \lambda\}$ and $a(\lambda) := \int_{\tilde{A}_\lambda} \delta^{\alpha_+ + \gamma} dx$. We write

$$(5.2) \quad \begin{aligned} \|g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|])\|_{L^1(\Omega; \delta^{\alpha_+ + \gamma})} &= \int_{\tilde{A}_1} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|]) \delta^{\alpha_+ + \gamma} dx \\ &\quad + \int_{\tilde{A}_1^c} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|]) \delta^{\alpha_+ + \gamma} dx \\ &\leq \int_{\tilde{A}_1} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|]) \delta^{\alpha_+ + \gamma} dx + g(1) \int_{\Omega} \delta^{\alpha_+ + \gamma} dx. \end{aligned}$$

We have

$$\int_{\tilde{A}_1} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|]) \delta^{\alpha_+ + \gamma} dx = a(1)g(1) + \int_1^\infty a(s)dg(s).$$

On the other hand, by (2.2) and Proposition 2.4 one gets, for every $s > 0$,

$$(5.3) \quad a(s) \leq C \left(\|\mathbb{K}_\mu[|\tilde{\nu}|]\|_{L_w^{p_{\mu,\gamma}^*, \gamma}(\Omega; \delta^{\alpha_+ + \gamma})}^{p_{\mu,\gamma}^*} + \|\mathbb{G}_\mu[|\tilde{\sigma}|]\|_{L_w^{q_{\mu,\gamma}^*, \gamma}(\Omega; \delta^{\alpha_+ + \gamma})}^{p_{\mu,\gamma}^*} \right) s^{-p_{\mu,\gamma}^*} \leq c_1 s^{-p_{\mu,\gamma}^*}$$

where $c_1 = c_1(N, \mu, \Omega, \gamma, \|\tilde{\nu}\|_{\mathfrak{M}(\partial\Omega)}, \|\tilde{\sigma}\|_{\mathfrak{M}(\Omega; \delta^{\alpha_+})})$. Thus

$$(5.4) \quad a(1)g(1) + \int_1^\infty a(s)dg(s) \leq C_1 + C_2 \int_1^\infty s^{-1-p_{\mu,\gamma}^*} g(s)ds \leq c_1 p_{\mu,\gamma}^* \Lambda_{g,\gamma}.$$

By combining the above estimates we obtain

$$\|g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|])\|_{L^1(\Omega; \delta^{\alpha_+ + \gamma})} \leq c_1 p_{\mu, \gamma}^* \Lambda_{g, \gamma} + g(1) \int_{\Omega} \delta^{\alpha_+ + \gamma} dx \leq c_2.$$

Similarly,

$$\|\tilde{g}(\mathbb{K}_\mu[|\nu|] + \mathbb{G}_\mu[|\tau|])\|_{L^1(\Omega; \delta^{\alpha_+ + \tilde{\gamma}})} \leq \tilde{c}_1 q_{\mu, \tilde{\gamma}}^* \Lambda_{\tilde{g}, \tilde{\gamma}} + \tilde{g}(1) \int_{\Omega} \delta^{\alpha_+ + \tilde{\gamma}} dx \leq \tilde{c}_2.$$

Thus (5.1) follows directly.

Step 2: Existence.

Put $u_0 := \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau]$. Let v_0 be the unique weak solution of the following problem

$$\begin{cases} -L_\mu v_0 + \delta^{\tilde{\gamma}} \tilde{g}(u_0) = \tilde{\tau} & \text{in } \Omega, \\ v_0 = \tilde{\nu} & \text{on } \partial\Omega. \end{cases}$$

For any $k \geq 1$, since g, \tilde{g} satisfy (5.1) there exist functions u_k and v_k satisfying

$$(5.5) \quad \begin{cases} -L_\mu u_k + \delta^\gamma g(v_{k-1}) = \tau & \text{in } \Omega, \\ -L_\mu v_k + \delta^{\tilde{\gamma}} \tilde{g}(u_k) = \tilde{\tau} & \text{in } \Omega, \\ u_k = \nu, \quad v_k = \tilde{\nu} & \text{on } \partial\Omega. \end{cases}$$

Moreover

$$(5.6) \quad \begin{aligned} u_k + \mathbb{G}_\mu[\delta^\gamma g(v_{k-1})] &= \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu], \\ v_k + \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(u_k)] &= \mathbb{G}_\mu[\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\nu}]. \end{aligned}$$

Since $g, \tilde{g} \geq 0$, it follows that, for every $k \geq 1$,

$$\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[\delta^\gamma g(\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}])] \leq u_k \leq \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] = u_0$$

and

$$\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau])] \leq v_k \leq \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}]$$

in Ω . Now, suppose that for some $k \geq 1$, $u_k \leq u_{k-1}$. Since g and \tilde{g} are nondecreasing, we deduce that

$$(5.7) \quad \begin{aligned} v_k &= \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(u_k)] \geq \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(u_{k-1})] = v_{k-1}, \\ u_{k+1} &= \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[\delta^\gamma g(v_k)] \leq \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[\delta^\gamma g(v_{k-1})] = u_k. \end{aligned}$$

This means that $\{v_k\}$ is nondecreasing and $\{u_k\}$ is nonincreasing. Hence, there exist u and v such that $u_k \downarrow u$ and $v_k \uparrow v$ in Ω and

$$\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[\delta^\gamma g(\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}])] \leq u \leq \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau],$$

$$\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau])] \leq v \leq \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}].$$

Since g and \tilde{g} are continuous and nondecreasing, we infer from monotone convergence theorem and (5.1) that $g(v_k) \rightarrow g(v)$ in $L^1(\Omega; \delta^{\gamma + \alpha_+})$ and $\tilde{g}(u_k) \rightarrow \tilde{g}(u)$ in $L^1(\Omega; \delta^{\tilde{\gamma} + \alpha_+})$. As a consequence,

$$\begin{aligned} \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(u_k)] &\rightarrow \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(u)] & \text{a.e. in } \Omega, \\ \mathbb{G}_\mu[\delta^\gamma g(v_k)] &\rightarrow \mathbb{G}_\mu[\delta^\gamma g(v)] & \text{a.e. in } \Omega. \end{aligned}$$

By letting $k \rightarrow \infty$ in (5.6), we obtain the desired result. \square

5.2. Source case: subcriticality. In this section for simplicity we consider system (1.33) with $\epsilon = 1$. Assume that $g(0) = \tilde{g}(0) = 0$. Put

$$(5.8) \quad \gamma^* = \min\{\alpha_+ + \gamma, \alpha_+ + \tilde{\gamma}, -\alpha_-\} > -1.$$

In preparation for proving Theorem F, we establish the following lemma:

Lemma 5.1. *Assume $\epsilon = 1$, g and \tilde{g} are bounded, nondecreasing and continuous functions in \mathbb{R} . Let $\tau, \tilde{\tau} \in \mathfrak{M}(\Omega; \delta^{\alpha_+})$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$. Assume there exist $a_1 > 0$, $b_1 > 0$ and $q_1 > 1$ such that (1.35) and (1.36) are satisfied. Then there exist $\lambda_*, \tilde{\lambda}_*, b_* > 0$ and $\varrho_* > 0$ depending on $N, \mu, \Omega, \gamma, \tilde{\gamma}, \Lambda_{\gamma, g}, \Lambda_{\tilde{\gamma}, \tilde{g}}, a_1, q_1$ such that the following holds. For every $b_1 \in (0, b_*)$ and $\sigma, \tilde{\sigma}, \varrho, \tilde{\varrho} \in (0, \varrho_*)$ the system*

$$(5.9) \quad \begin{cases} -L_\mu u = \delta^\gamma g(v + \tilde{\varrho} \mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}]) & \text{in } \Omega, \\ -L_\mu v = \delta^{\tilde{\gamma}} \tilde{g}(u + \varrho \mathbb{K}_\mu[\nu] + \sigma \mathbb{G}_\mu[\tau]) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

admits a weak solution (u, v) satisfying

$$(5.10) \quad \begin{aligned} \|u\|_{L_w^{p_{\mu, \tilde{\gamma}}^*}(\Omega; \delta^{\alpha_+ + \tilde{\gamma}})} + \|u\|_{L^{q_1}(\Omega; \delta^{\gamma^*})} &\leq \lambda_*, \\ \|v\|_{L_w^{p_{\mu, \gamma}^*}(\Omega; \delta^{\alpha_+ + \gamma})} + \|v\|_{L^{q_1}(\Omega; \delta^{\gamma^*})} &\leq \tilde{\lambda}_*. \end{aligned}$$

Proof. Without loss of generality, we assume that $\|\tau\|_{\mathfrak{M}(\Omega; \delta^{\alpha_+})} = \|\tilde{\tau}\|_{\mathfrak{M}(\Omega; \delta^{\alpha_+})} = \|\nu\|_{\mathfrak{M}(\partial\Omega)} = \|\tilde{\nu}\|_{\mathfrak{M}(\partial\Omega)} = 1$. We shall use Schauder fixed point theorem to show the existence of positive solutions of (5.9).

Define

$$(5.11) \quad \begin{aligned} \mathbb{S}(w) &:= \mathbb{G}_\mu[\delta^\gamma g(w + \tilde{\varrho} \mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}])], \\ \tilde{\mathbb{S}}(w) &:= \mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(w + \varrho \mathbb{K}_\mu[\nu] + \sigma \mathbb{G}_\mu[\tau])], \quad \forall w \in L^1(\Omega). \end{aligned}$$

Set

$$\begin{aligned} \mathbf{M}_1(w) &:= \|w\|_{L_w^{p_{\mu, \gamma}^*}(\Omega; \delta^{\alpha_+ + \gamma})}, \quad \forall w \in L_w^{p_{\mu, \gamma}^*}(\Omega; \delta^{\alpha_+ + \gamma}), \\ \tilde{\mathbf{M}}_1(w) &:= \|w\|_{L_w^{p_{\mu, \tilde{\gamma}}^*}(\Omega; \delta^{\alpha_+ + \tilde{\gamma}})}, \quad \forall w \in L_w^{p_{\mu, \tilde{\gamma}}^*}(\Omega; \delta^{\alpha_+ + \tilde{\gamma}}), \\ \mathbf{M}_2(w) &= \|w\|_{L^{q_1}(\Omega; \delta^{\gamma^*})}, \quad \forall w \in L^{q_1}(\Omega; \delta^{\gamma^*}), \\ \mathbf{M}(w) &= \mathbf{M}_1(w) + \mathbf{M}_2(w), \quad \forall w \in L_w^{p_{\mu, \gamma}^*}(\Omega; \delta^{\alpha_+ + \gamma}) \cap L^{q_1}(\Omega; \delta^{\gamma^*}), \\ \tilde{\mathbf{M}}(w) &= \tilde{\mathbf{M}}_1(w) + \mathbf{M}_2(w), \quad \forall w \in L_w^{p_{\mu, \tilde{\gamma}}^*}(\Omega; \delta^{\alpha_+ + \tilde{\gamma}}) \cap L^{q_1}(\Omega; \delta^{\gamma^*}). \end{aligned}$$

Step 1: Upper bound for $g(w + \tilde{\varrho} \mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}])$ in $L^1(\Omega; \delta^{\alpha_+ + \gamma})$ with $w \in L_w^{p_{\mu, \gamma}^*}(\Omega; \delta^{\alpha_+ + \gamma}) \cap L^{q_1}(\Omega; \delta^{\gamma^*})$.

For $\lambda > 0$, set $\tilde{B}_\lambda := \{x \in \Omega : |w| + \tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|] + \tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|] > \lambda\}$ and $b(\lambda) := \int_{\tilde{B}_\lambda} \delta^{\alpha_++\gamma} dx$. We write

$$\begin{aligned}
 \|g(w + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])\|_{L^1(\Omega; \delta^{\alpha_++\gamma})} &\leq \int_{\tilde{B}_1} g(|w| + \tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|] + \tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|]) \delta^{\alpha_++\gamma} dx \\
 &\quad + \int_{\tilde{B}_1^c} g(|w| + \tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|] + \tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|]) \delta^{\alpha_++\gamma} dx \\
 (5.12) \quad &\quad - \int_{\tilde{B}_1} g(-|w| - \tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|] - \tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|]) \delta^{\alpha_++\gamma} dx \\
 &\quad - \int_{\tilde{B}_1^c} g(-|w| - \tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|] - \tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|]) \delta^{\alpha_++\gamma} dx \\
 &=: I + II + III + IV.
 \end{aligned}$$

We first estimate I . Since $g \in C(\mathbb{R}_+)$ is nondecreasing, one gets

$$I = b(1)g(1) + \int_1^\infty b(s)dg(s).$$

Since g is bounded, there exists an increasing sequence of real positive number $\{\ell_j\}$ such that

$$(5.13) \quad \lim_{j \rightarrow \infty} \ell_j = \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} \ell_j^{-p_{\mu,\gamma}^*} g(\ell_j) = 0.$$

Observe that

$$\int_1^\infty b(s)dg(s) = \lim_{j \rightarrow \infty} \int_1^{\ell_j} b(s)dg(s).$$

On the other hand, by (2.2) one gets, for every $s > 0$,

$$(5.14) \quad a(s) \leq \| |w| + \tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|] + \tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|] \|_{L_w^{p_{\mu,\gamma}^*}(\Omega; \delta^{\alpha_++\gamma})}^{p_{\mu,\gamma}^*} s^{-p_{\mu,\gamma}^*} \leq c_{31}(\mathbf{M}_1(w) + \tilde{\varrho} + \tilde{\sigma})^{p_{\mu,\gamma}^*} s^{-p_{\mu,\gamma}^*}$$

where $c_{31} = c_{31}(N, \mu, \Omega)$. Using (5.14), we obtain

$$\begin{aligned}
 b(1)g(1) + \int_1^{\ell_j} b(s)dg(s) &\leq c_{31}(\mathbf{M}_1(w) + \tilde{\varrho} + \tilde{\sigma})^{p_{\mu,\gamma}^*} g(1) + c_{31}(\mathbf{M}_1(w) + \tilde{\varrho} + \tilde{\sigma})^{p_{\mu,\gamma}^*} \int_1^{\ell_j} s^{-p_{\mu,\gamma}^*} dg(s) \\
 &\leq c_{31}(\mathbf{M}_1(w) + \tilde{\varrho} + \tilde{\sigma})^{p_{\mu,\gamma}^*} \ell_j^{-p_{\mu,\gamma}^*} g(\ell_j) + c_{31}p_{\mu,\gamma}^*(\mathbf{M}_1(w) + \tilde{\varrho} + \tilde{\sigma})^{p_{\mu,\gamma}^*} \int_1^{\ell_j} s^{-1-p_{\mu,\gamma}^*} g(s) ds.
 \end{aligned}$$

By virtue of (5.13), letting $j \rightarrow \infty$ yields

$$(5.15) \quad I \leq c_{31}p_{\mu,\gamma}^*(\mathbf{M}_1(w) + \tilde{\varrho} + \tilde{\sigma})^{p_{\mu,\gamma}^*} \int_1^\infty s^{-1-p_{\mu,\gamma}^*} g(s) ds.$$

Similarly we have

$$III \leq -c_{31}p_{\mu,\gamma}^*(\mathbf{M}_1(w) + \tilde{\varrho} + \tilde{\sigma})^{p_{\mu,\gamma}^*} \int_1^\infty s^{-1-p_{\mu,\gamma}^*} g(-s) ds.$$

To handle the remaining terms II, III , without loss of generality, we assume $q_1 \in (1, \frac{N+\gamma^*}{N+\alpha_+-2})$. Since g satisfies condition (1.35), it follows that

$$\begin{aligned}
 \max\{II, IV\} &\leq a_1 \int_{\tilde{B}_1^c} (|w| + \tilde{\varrho} \mathbb{K}_\mu[|\tilde{\nu}|] + \tilde{\sigma} \mathbb{K}_\mu[|\tilde{\tau}|])^{q_1} \delta^{\alpha_++\gamma} dx + b_1 \int_{\tilde{B}_1^c} \delta^{\alpha_++\gamma} dx \\
 (5.16) \quad &\leq a_1 c_{34} \int_{\Omega} |w|^{q_1} \delta^{\alpha_++\gamma} dx + a_1 c_{34} (\tilde{\varrho}^{q_1} + \tilde{\sigma}^{q_1}) + c_{34} b_1 \\
 &\leq a_1 c_{35} \mathbf{M}_2(w)^{q_1} + a_1 c_{34} (\tilde{\varrho}^{q_1} + \tilde{\sigma}^{q_1}) + c_{34} b_1
 \end{aligned}$$

where $c_i = c_i(N, \mu, \Omega)$, $i = 33, 34$.

Combining (5.12), (5.15) and (5.16) yields

$$(5.17) \quad \|g(w + \tilde{\varrho} \mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}])\|_{L^1(\Omega; \delta^{\alpha_++\gamma})} \leq c_{33} \Lambda_{g,\gamma} \mathbf{M}_1(w)^{p_{\mu,\gamma}^*} + c_{35} a_1 \mathbf{M}_2(w)^{q_1} + c_{34} b_1 + d_{\tilde{\varrho}, \tilde{\sigma}}$$

where $d_{\tilde{\varrho}, \tilde{\sigma}} = c_{33} \Lambda_{g,\gamma} (\tilde{\varrho}^{p_{\mu,\gamma}^*} + \tilde{\sigma}^{p_{\mu,\gamma}^*}) + c_{34} a_1 (\tilde{\varrho}^{q_1} + \tilde{\sigma}^{q_1})$.

Step 2: Estimates on $\mathbf{M}_1, \mathbf{M}_2$ and \mathbf{M} .

From (2.11), we have

$$\begin{aligned}
 \tilde{\mathbf{M}}_1(\mathbb{S}(w)) &= \|\mathbb{G}_\mu[\delta^\gamma g(w + \tilde{\varrho} \mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}])]\|_{L_w^{p_{\mu,\tilde{\gamma}}^*}(\Omega; \delta^{\alpha_++\gamma})} \\
 (5.18) \quad &\leq c_7 \|g(w + \tilde{\varrho} \mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}])\|_{L^1(\Omega; \delta^{\alpha_++\gamma})}.
 \end{aligned}$$

It follows that

$$(5.19) \quad \tilde{\mathbf{M}}_1(\mathbb{S}(w)) \leq c_7 c_{33} \Lambda_{g,\gamma} \mathbf{M}_1(w)^{p_{\mu,\gamma}^*} + c_7 c_{35} a_1 \mathbf{M}_2(w)^{q_1} + c_7 c_{34} b_1 + c_7 d_{\tilde{\varrho}, \tilde{\sigma}}.$$

Applying (2.11), we get

$$\begin{aligned}
 \mathbf{M}_2(\mathbb{S}(w)) &= \|\mathbb{G}_\mu[\delta^{\tilde{\gamma}} \tilde{g}(w + \varrho \mathbb{K}_\mu[\nu] + \sigma \mathbb{G}_\mu[\tau])]\|_{L^{q_1}(\Omega; \delta^{\gamma^*})} \\
 &\leq c_{36} \|\tilde{g}(w + \varrho \mathbb{K}_\mu[\nu] + \sigma \mathbb{G}_\mu[\tau])\|_{L^1(\Omega; \delta^{\alpha_++\gamma})},
 \end{aligned}$$

which implies

$$(5.20) \quad \mathbf{M}_2(\mathbb{S}(w)) \leq c_{36} c_{33} \Lambda_{g,\gamma} \mathbf{M}_1(w)^{p_{\mu,\gamma}^*} + c_{36} c_{35} a_1 \mathbf{M}_2(w)^{q_1} + c_{36} c_{34} b_1 + c_{36} d_{\tilde{\varrho}, \tilde{\sigma}}.$$

Consequently,

$$(5.21) \quad \tilde{\mathbf{M}}(\mathbb{S}(w)) \leq c_{37} \Lambda_{g,\gamma} \mathbf{M}_1(w)^{p_{\mu,\gamma}^*} + c_{38} a_1 \mathbf{M}_2(w)^{q_1} + c_{40} b_1 + c_{39} d_{\tilde{\varrho}, \tilde{\sigma}}$$

where $c_{37} = (c_7 + c_{36})c_{33}$, $c_{38} = (c_7 + c_{36})c_{35}$, $c_{39} = c_7 + c_{36}$, $c_{40} = (c_7 + c_{36})c_{34}$. Similarly, we can show that

$$(5.22) \quad \mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \tilde{c}_{37} \Lambda_{\tilde{g}, \tilde{\gamma}} \tilde{\mathbf{M}}_1(w)^{p_{\mu,\tilde{\gamma}}^*} + \tilde{c}_{38} a_1 \mathbf{M}_2(w)^{q_1} + \tilde{c}_{40} b_1 + \tilde{c}_{39} d_{\varrho, \sigma}$$

where \tilde{c}_i ($37 \leq i \leq 40$) are positive constants. Define the functions η and $\tilde{\eta}$ as follows

$$\begin{aligned}
 \eta(\lambda) &:= \max\{c_{37} \Lambda_{g,\gamma}, \tilde{c}_{37} \Lambda_{\tilde{g}, \tilde{\gamma}}\} \lambda^{p_{\mu,\gamma}^*} + \max\{c_{38}, \tilde{c}_{38}\} a_1 \lambda^{q_1} + \max\{c_{40}, \tilde{c}_{40}\} b_1 + \max\{c_{39} d_{\varrho}, \tilde{c}_{39} \tilde{d}_{\varrho}\} \\
 \tilde{\eta}(\lambda) &:= \max\{c_{37} \Lambda_{g,\gamma}, \tilde{c}_{37} \Lambda_{\tilde{g}, \tilde{\gamma}}\} \lambda^{p_{\mu,\tilde{\gamma}}^*} + \max\{c_{38}, \tilde{c}_{38}\} a_1 \lambda^{q_1} + \max\{c_{40}, \tilde{c}_{40}\} b_1 + \max\{c_{39} d_{\varrho}, \tilde{c}_{39} \tilde{d}_{\varrho}\}.
 \end{aligned}$$

By (5.21) and (5.22), we deduce

$$\tilde{\mathbf{M}}(\mathbb{S}(w)) \leq \eta(\mathbf{M}(w)) \quad \text{and} \quad \mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \tilde{\eta}(\tilde{\mathbf{M}}(w)).$$

Since $p_{\mu,\gamma}^* > 1$, $p_{\mu,\tilde{\gamma}}^* > 1$ and $q_1 > 1$, there exist $\varrho_* > 0$ and $b_* > 0$ depending on $N, \mu, \Omega, \gamma, \tilde{\gamma}, \Lambda_{g,\gamma}, \Lambda_{\tilde{g}, \tilde{\gamma}}, a_1, q_1$ such that for any $\varrho, \tilde{\varrho} \in (0, \varrho_*)$ and $b \in (0, b_*)$ there exist $\lambda_* > 0$ and $\tilde{\lambda}_* > 0$ such that

$$\eta(\lambda_*) = \tilde{\lambda}_* \quad \text{and} \quad \tilde{\eta}(\tilde{\lambda}_*) = \lambda_*.$$

Here λ_* and $\tilde{\lambda}_*$ depend on $N, \mu, \Omega, \gamma, \tilde{\gamma}, \Lambda_{\gamma,g}, \Lambda_{\tilde{\gamma},\tilde{g}}, a_1, q_1$. Therefore,

$$(5.23) \quad \begin{aligned} \mathbf{M}(w) \leq \lambda_* &\implies \tilde{\mathbf{M}}(\mathbb{S}(w)) \leq \tilde{\lambda}_* \\ \tilde{\mathbf{M}}(w) \leq \tilde{\lambda}_* &\implies \mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \lambda_*. \end{aligned}$$

Step 3: We apply Schauder fixed point theorem to our setting.

For $w \in L^1(\Omega)$, put

$$(5.24) \quad \mathbb{T}(w_1, w_2) = (\mathbb{S}(w_2), \tilde{\mathbb{S}}(w_1)).$$

Set

$$\mathcal{D} = \{(\varphi, \tilde{\varphi}) \in L^1_+(\Omega) \times L^1_+(\Omega) : \mathbf{M}(\varphi) \leq \lambda_* \text{ and } \tilde{\mathbf{M}}(\tilde{\varphi}) \leq \tilde{\lambda}_*\}.$$

Clearly, \mathcal{D} is a convex subset of $L^1(\Omega) \times L^1(\Omega)$. We shall show that \mathcal{D} is a closed subset of $(L^1(\Omega))^2$. Indeed, let $\{(\varphi_m, \tilde{\varphi}_m)\}$ be a sequence in \mathcal{D} converging to $(\varphi, \tilde{\varphi})$ in $(L^1(\Omega))^2$. Obviously, $\varphi \geq 0$ and $\tilde{\varphi} \geq 0$. We can extract a subsequence, still denoted by the same notation, such that $(\varphi_m, \tilde{\varphi}_m) \rightarrow (\varphi, \tilde{\varphi})$ a.e. in Ω . Consequently, by Fatou's lemma,

$$\mathbf{M}_i(\varphi) \leq \liminf_{m \rightarrow \infty} \mathbf{M}_i(\varphi_m), \quad \mathbf{M}_i(\tilde{\varphi}) \leq \liminf_{m \rightarrow \infty} \mathbf{M}_i(\tilde{\varphi}_m)$$

for $i = 1, 2$. It follows that $\mathbf{M}(\varphi) \leq \lambda_*$ and $\mathbf{M}(\tilde{\varphi}) \leq \tilde{\lambda}_*$. So $(\varphi, \tilde{\varphi}) \in \mathcal{D}$ and therefore \mathcal{D} is a closed subset of $L^1(\Omega) \times L^1(\Omega)$.

Clearly, \mathbb{T} is well defined in \mathcal{D} . For $(w, \tilde{w}) \in \mathcal{D}$, we get $\mathbf{M}(w) \leq \lambda_*$ and $\mathbf{M}(\tilde{w}) \leq \tilde{\lambda}_*$, hence $\tilde{\mathbf{M}}(\mathbb{S}(w)) \leq \tilde{\lambda}_*$ and $\mathbf{M}(\tilde{\mathbb{S}}(\tilde{w})) \leq \lambda_*$. It follows that $\mathbb{T}(\mathcal{D}) \subset \mathcal{D}$.

We observe that \mathbb{T} is continuous. Indeed, if $w_m \rightarrow w$ and $\tilde{w}_m \rightarrow \tilde{w}$ as $m \rightarrow \infty$ in $L^1(\Omega)$ then since $g, \tilde{g} \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, it follows that

$$g(\tilde{w}_m + \varrho \mathbb{K}_\mu[\nu] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}]) \rightarrow g(\tilde{w} + \varrho \mathbb{K}_\mu[\nu] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}]) \quad \text{in } L^1(\Omega; \delta^{\alpha+\gamma}),$$

and

$$\tilde{g}(w_m + \varrho \mathbb{K}_\mu[\nu] + \sigma \mathbb{G}_\mu[\tau]) \rightarrow \tilde{g}(w + \varrho \mathbb{K}_\mu[\nu] + \sigma \mathbb{G}_\mu[\tau]) \quad \text{in } L^1(\Omega; \delta^{\alpha+\tilde{\gamma}})$$

as $m \rightarrow \infty$. By (2.11), $\mathbb{S}(\tilde{w}_m) \rightarrow \mathbb{S}(\tilde{w})$ and $\tilde{\mathbb{S}}(w_m) \rightarrow \tilde{\mathbb{S}}(w)$ as $m \rightarrow \infty$ in $L^1(\Omega)$. Thus $\mathbb{T}(w_m, \tilde{w}_m) \rightarrow \mathbb{T}(w, \tilde{w})$ in $L^1(\Omega) \times L^1(\Omega)$.

We next show that \mathbb{T} is a compact operator. Let $\{(w_m, \tilde{w}_m)\} \subset \mathcal{D}$ and for each $m \geq 1$, put $\psi_m = \mathbb{S}(\tilde{w}_m)$ and $\tilde{\psi}_m = \tilde{\mathbb{S}}(w_m)$. Hence $\{\Delta \psi_m\}$ and $\{\Delta \tilde{\psi}_m\}$ are uniformly bounded in $L^p(G)$ for every subset $G \Subset \Omega$. Therefore $\{\psi_m\}$ is uniformly bounded in $W^{1,p}(G)$. Consequently, there exists a subsequence, still denoted by the same notation, and functions $\psi, \tilde{\psi}$ such that $(\psi_m, \tilde{\psi}_m) \rightarrow (\psi, \tilde{\psi})$ a.e. in Ω . By dominated convergence theorem, $(\psi_m, \tilde{\psi}_m) \rightarrow (\psi, \tilde{\psi})$ in $L^1(\Omega) \times L^1(\Omega)$. Thus \mathbb{T} is compact.

By Schauder fixed point theorem there is $(u, v) \in \mathcal{D}$ such that $\mathbb{T}(u, v) = (u, v)$. If $\mu \in (0, C_H)$, due to Proposition 2.6, $\text{tr}^*(u) = \text{tr}^*(v) = 0$ and the result follows. \square

Proof of Theorem F.I. Let $\{g_n\}$ and $\{\tilde{g}_n\}$ be the sequences of continuous, nondecreasing functions defined on \mathbb{R}_+ such that

$$(5.25) \quad \begin{aligned} g_n(0) = g(0), |g_n| \leq |g_{n+1}| \leq |g|, \sup_{\mathbb{R}_+} |g_n| = n \text{ and } \lim_{n \rightarrow \infty} \|g_n - g\|_{L^\infty_{\text{loc}}(\mathbb{R}_+)} = 0, \\ \tilde{g}_n(0) = \tilde{g}(0), |\tilde{g}_n| \leq |\tilde{g}_{n+1}| \leq |\tilde{g}|, \sup_{\mathbb{R}_+} |\tilde{g}_n| = n \text{ and } \lim_{n \rightarrow \infty} \|\tilde{g}_n - \tilde{g}\|_{L^\infty_{\text{loc}}(\mathbb{R}_+)} = 0. \end{aligned}$$

Due to Lemma 5.1, there exists $\lambda_*, \tilde{\lambda}_*, b_* > 0$ and $\varrho_* > 0$ depending on $N, \mu, \Omega, \gamma, \tilde{\gamma}, \Lambda_{\gamma,g}, \Lambda_{\tilde{\gamma},\tilde{g}}, a_1, q_1$ such that for every $b_1 \in (0, b_*)$, $\tilde{\varrho}, \varrho \in (0, \varrho_*)$ and $n \geq 1$ there exists a solution

$(w_n, \tilde{w}_n) \in \mathcal{D}$ of

$$(5.26) \quad \begin{cases} -L_\mu w_n = \delta^\gamma g_n(\tilde{w}_n + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]) & \text{in } \Omega, \\ -L_\mu \tilde{w}_n = \delta^{\tilde{\gamma}} \tilde{g}_n(w_n + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]) & \text{in } \Omega, \\ w_n = \tilde{w}_n = 0 & \text{on } \Omega. \end{cases}$$

For each n , set $u_n = w_n + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]$ and $v_n = \tilde{w}_n + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]$. Then

$$(5.27) \quad -\int_\Omega u_n L_\mu \phi dx = \int_\Omega \delta^\gamma g_n(v_n) \phi dx + \sigma \int_\Omega \phi d\tau - \varrho \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \phi dx, \quad \forall \phi \in \mathbf{X}(\Omega),$$

$$(5.28) \quad -\int_\Omega v_n L_\mu \phi dx = \int_\Omega \delta^{\tilde{\gamma}} \tilde{g}_n(u_n) \phi dx + \tilde{\sigma} \int_\Omega \phi d\tilde{\tau} - \tilde{\varrho} \int_\Omega \mathbb{K}_\mu[\tilde{\nu}] L_\mu \phi dx, \quad \forall \phi \in \mathbf{X}(\Omega).$$

Since $\{(w_n, \tilde{w}_n)\} \subset \mathcal{D}$ and the fact that $\Lambda_{g_n, \gamma} \leq \Lambda_{g, \gamma}$, we obtain from (5.17) that

$$(5.29) \quad \|g_n(v_n)\|_{L^1(\Omega; \delta^{\alpha_+ + \gamma})} \leq c_{33} \Lambda_{g, \gamma} \lambda_*^{p_{\mu, \gamma}^*} + c_{35} a_1 \lambda_*^{q_1} + c_{34} b_* + d_{\varrho_*}$$

Hence the sequence $\{g(v_n)\}$ is uniformly bounded in $L^1(\Omega; \delta^{\alpha_+ + \gamma})$. Since $\{(w_n, \tilde{w}_n)\} \subset \mathcal{D}$, the sequence $\{\frac{\mu}{\delta^2} w_n\}$ and $\{\frac{\mu}{\delta^2} \tilde{w}_n\}$ are uniformly bounded in $L^{q_1}(G)$ for every subset $G \Subset \Omega$. As a consequence, $\{\Delta w_n\}$ and $\{\Delta \tilde{w}_n\}$ are uniformly bounded in $L^1(G)$ for every subset $G \Subset \Omega$. By regularity result for elliptic equations, there exists a subsequence, still denoted by the same notations, and functions w and \tilde{w} such that $(w_n, \tilde{w}_n) \rightarrow (w, \tilde{w})$ a.e. in Ω . Therefore $(u_n, v_n) \rightarrow (u, v)$ a.e. in Ω with $u = w + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]$ and $u = \tilde{w} + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]$. Moreover $(\tilde{g}_n(u_n), g_n(v_n)) \rightarrow (\tilde{g}(u), g(v))$ a.e. in Ω .

We show that $u_n \rightarrow u$ in $L^1(\Omega; \delta^{\alpha_+})$. Since $\{w_n\}$ is uniformly bounded in $L^{q_1}(\Omega; \delta^{\gamma^*})$, by (2.12), we derive that $\{u_n\}$ is uniformly bounded in $L^{q_1}(\Omega; \delta^{\alpha_+})$. Due to Holder inequality, $\{u_n\}$ is uniformly integrable with respect to $\delta^{\alpha_+} dx$. We invoke Vitali's convergence theorem to derive that $u_n \rightarrow u$ in $L^1(\Omega; \delta^{\alpha_+})$. Similarly, one can prove that $v_n \rightarrow v$ in $L^1(\Omega; \delta^{\alpha_+})$.

We next prove that $g_n(v_n) \rightarrow g(v)$ in $L^1(\Omega; \delta^{\alpha_+ + \gamma})$. For $\lambda > 0$ and $n \in \mathbb{N}$ set $B_{n, \lambda} := \{x \in \Omega : |v_n| > \lambda\}$ and $b_n(\lambda) := \int_{B_{n, \lambda}} \delta^{\alpha_+ + \gamma} dx$. For any Borel set $E \subset \Omega$,

$$(5.30) \quad \begin{aligned} \int_E g_n(v_n) \delta^{\alpha_+ + \gamma} dx &= \int_{E \cap B_{n, \lambda}} g_n(v_n) \delta^{\alpha_+ + \gamma} dx + \int_{E \cap B_{n, \lambda}^c} g_n(v_n) \delta^{\alpha_+ + \gamma} dx \\ &\leq \int_{B_{n, \lambda}} g_n(v_n) \delta^{\alpha_+ + \gamma} dx + m_{g, \lambda} \int_E \delta^{\alpha_+ + \gamma} dx \\ &\leq b_n(\lambda) g_n(\lambda) + \int_\lambda^\infty b_n(s) dg_n(s) + m_{g, \lambda} \int_E \delta^{\alpha_+ + \gamma} dx. \end{aligned}$$

where $m_{g, \lambda} := \sup_{[0, \lambda]} g$. By proceeding as in the proof of Lemma 5.1 in order to get (5.15), we deduce

$$(5.31) \quad b_n(\lambda) g_n(\lambda) + \int_\lambda^\infty b_n(s) dg_n(s) \leq c_{41} \int_\lambda^\infty s^{-1-p_{\mu, \gamma}^*} g_n(s) ds \leq c_{41} \int_\lambda^\infty s^{-1-p_{\mu, \gamma}^*} g(s) ds$$

where c_{41} depends on $N, \mu, \gamma, \tilde{\gamma}, \Lambda_{\gamma, g}, \Lambda_{\tilde{\gamma}, \tilde{g}}, a_1, q_1$. Note that the term on the right hand-side of (5.31) tends to 0 as $\lambda \rightarrow \infty$. Take arbitrarily $\varepsilon > 0$, there exists $\lambda > 0$ such that the right hand-side of (5.31) is smaller than $\frac{\varepsilon}{2}$. Fix such λ and put $\eta = \frac{\varepsilon}{2m_{g, \lambda}}$. Then, by (5.30),

$$\int_E \delta(x)^{\alpha_+ + \gamma} dx \leq \eta \implies \int_E g_n(v_n) \delta(x)^{\alpha_+ + \gamma} dx < \varepsilon.$$

Therefore the sequence $\{g_n(v_n)\}$ is uniformly integrable with respect to $\delta^{\alpha_+ + \gamma} dx$. Due to Vitali convergence theorem, we deduce that $g_n(v_n) \rightarrow g(v)$ in $L^1(\Omega; \delta^{\alpha_+ + \gamma})$.

By sending $n \rightarrow \infty$ in each term of (5.27) we obtain

$$(5.32) \quad - \int_{\Omega} u L_{\mu} \phi dx = \int_{\Omega} \delta^{\gamma} g(v) \phi dx + \sigma \int_{\Omega} \phi d\tau - \varrho \int_{\Omega} \mathbb{K}_{\mu}[\nu] L_{\mu} \phi dx, \quad \forall \phi \in \mathbf{X}(\Omega).$$

Similarly, one can show that $\tilde{g}_n(u_n) \rightarrow \tilde{g}(u)$ in $L^1(\Omega; \delta^{\alpha_+ + \tilde{\gamma}})$. By letting $n \rightarrow \infty$ in (5.28), we get

$$(5.33) \quad - \int_{\Omega} v L_{\mu} \phi dx = \int_{\Omega} \delta^{\tilde{\gamma}} \tilde{g}(u) \phi dx + \tilde{\sigma} \int_{\Omega} \phi d\tilde{\tau} - \tilde{\varrho} \int_{\Omega} \mathbb{K}_{\mu}[\tilde{\nu}] L_{\mu} \phi dx, \quad \forall \phi \in \mathbf{X}(\Omega).$$

Thus (u, v) is a solution of (1.24). Moreover, if $\mu \in (0, C_H)$ then $\text{tr}^*(u) = \varrho\nu$ and $\text{tr}^*(v) = \tilde{\varrho}\tilde{\nu}$. \square

5.3. Source case : sublinearity. We next deal with the case where g and \tilde{g} are sublinear.

Proof of Theorem F.II. The proof is similar to that of Lemma 5.1, also based on Schauder fixed point theorem. So we point out only the main modifications. Let \mathbb{S} and $\tilde{\mathbb{S}}$ be the operators defined in (5.11). Put

$$\mathbf{N}_1(w) := \|w\|_{L^{q_1}(\Omega; \delta^{\gamma^*})}, \quad \forall w \in L^{q_1}(\Omega; \delta^{\gamma^*}),$$

and

$$\mathbf{N}_2(w) := \|w\|_{L^1(\Omega; \delta^{\gamma^*})}, \quad \forall w \in L^1(\Omega; \delta^{\gamma^*}).$$

Combining (2.11), (2.12) and (1.37) leads to

$$\begin{aligned} \mathbf{N}_2(\mathbb{S}(w)) &\leq c_{42} \|g(w + \tilde{\varrho}\mathbb{K}_{\mu}[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_{\mu}[\tilde{\tau}])\|_{L^1(\Omega; \delta^{\alpha_+ + \gamma})} \\ &\leq c_{42} \int_{\Omega} a_2(|w| + \tilde{\varrho}\mathbb{K}_{\mu}[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_{\mu}[\tilde{\tau}])^{q_1} \delta^{\alpha_+ + \gamma} dx + c_{42} b_2 \int_{\Omega} \delta^{\alpha_+ + \gamma} dx \\ &\leq c_{42} a_2 \int_{\Omega} |w|^{q_1} \delta^{\alpha_+ + \gamma} dx + c_{43} (\varrho^{q_1} + \tilde{\sigma}^{q_1} + b_2) \\ &\leq c_{44} a_2 \mathbf{N}_1(w)^{q_1} + c_{43} (\tilde{\varrho}^{q_1} + \tilde{\sigma}^{q_1} + b_2) \end{aligned}$$

where $c_i = c_i(N, \mu, \Omega, \gamma, \tilde{\gamma}, q_2, q)$ ($i = 42, 43, 44$).

On the other hand

$$\begin{aligned} \mathbf{N}_1(\tilde{\mathbb{S}}(w)) &\leq c_{42} \left(\int_{\Omega} a_2^{q_1} (|w| + \varrho\mathbb{K}_{\mu}[\nu] + \sigma\mathbb{G}_{\mu}[\tau])^{q_1 q_2} \delta^{\alpha_+ + \gamma} dx \right)^{\frac{1}{q_1}} + c_{42} b_2 \int_{\Omega} \delta^{\alpha_+ + \gamma} dx \\ &\leq c_{42} a_2 \left(\int_{\Omega} |w| \delta^{\alpha_+ + \gamma} dx \right)^{q_2} + c_{43} (\varrho^{q_2} + \sigma^{q_2} + b_2) \\ &\leq c_{44} a_2 \mathbf{N}_2(w)^{q_2} + c_{43} (\varrho^{q_2} + \sigma^{q_2} + b_2). \end{aligned}$$

Define

$$\xi_1(\lambda) := c_{44} a_2 \lambda^{q_1} + c_{43} (\tilde{\varrho}^{q_1} + \tilde{\sigma}^{q_1} + b_2),$$

and

$$\xi_2(\lambda) := c_{44} a_2 \lambda^{q_2} + c_{43} (\varrho^{q_2} + \sigma^{q_2} + b_2).$$

Then

$$\mathbf{N}_2(\mathbb{S}(w)) \leq \xi_1(\mathbf{N}_1(w)) \quad \text{and} \quad \mathbf{N}_1(\tilde{\mathbb{S}}(w)) \leq \xi_2(\mathbf{N}_2(w)),$$

If $q_1 q_2 < 1$ then we can find λ_1 and λ_2 such that $\xi_1(\lambda_1) = \lambda_2$ and $\xi_2(\lambda_2) = \lambda_1$. Thus if $\mathbf{N}_1(w_2) < \lambda_2$, then $\mathbf{N}_2(\mathbb{S}(w_2)) < \lambda_1$ and if $\mathbf{N}_2(w_1) < \lambda_1$, then $\mathbf{N}_1(\tilde{\mathbb{S}}(w_1)) < \lambda_2$.

If $q_1 q_2 = 1$ and a_2 small enough we can find, λ_1 and λ_2 such that $\xi_1(\lambda_1) = \lambda_2$ and $\xi_2(\lambda_2) = \lambda_1$.

The rest of the proof can be proceeded as in the proof of Lemma 5.1 and the proof of Theorem F.I. and we omit it. \square

5.4. Source case : subcriticality and sublinearity.

Proof of Theorem F.III.

Set

$$\mathbf{N}(w) := \|w\|_{L^{q_1}(\Omega; \delta^{\gamma^*})}, \quad \forall w \in L^{q_1}(\Omega; \delta^{\gamma^*}).$$

By an argument similar to the proof of Lemma 5.1 and and Theorem F.II, we get

$$\begin{aligned} \mathbf{N}(\mathbb{S}(w)) &\leq \|g(w + \rho \mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma} \mathbb{G}_\mu[\tilde{\tau}])\|_{L^1(\Omega, \delta^{\alpha_+ + \gamma})} \\ &\leq c_{33} \Lambda_{g, \gamma} \mathbf{M}_1(w)^{p_{\mu, \gamma}^*} + c_{35} a_1 \mathbf{M}_2(w)^{q_1} + c_{34} b_1 + d_{\tilde{\varrho}, \tilde{\sigma}} \\ &\leq c_{33} \Lambda_{g, \gamma} \mathbf{M}_1(w)^{p_{\mu, \gamma}^*} + c_{35} a_1 \mathbf{M}_2(w)^{q_1} + c_{34} b_1 + d_{\tilde{\varrho}, \tilde{\sigma}}. \end{aligned}$$

On the other hand

$$\begin{aligned} \mathbf{M}(\tilde{\mathbb{S}}(w)) &\leq 2 \|\tilde{g}(w + \rho \mathbb{K}_\mu[\nu] + \sigma \mathbb{G}_\mu[\tau])\|_{L^1(\Omega, \delta^{\alpha_+ + \tilde{\gamma}})} \\ &\leq 2c_{44} a_2 \mathbf{N}(w)^{q_2} + 2c_{43} (\varrho^{q_2} + b_2). \end{aligned}$$

Set

$$\begin{aligned} \hat{\xi}_1(\lambda) &:= c_{33} \Lambda_{g, \gamma} \lambda^{p_{\mu, \gamma}^*} + c_{35} a_1 \lambda^{q_1} + c_{34} b_1 + d_\rho, \\ \hat{\xi}_2(\lambda) &:= 2c_{44} a_2 \lambda^{q_2} + 2c_{43} (\rho^{q_2} + b_2). \end{aligned}$$

Then

$$\mathbf{N}(\mathbb{S}(w)) \leq \hat{\xi}_1(\mathbf{M}(w)) \quad \text{and} \quad \mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \hat{\xi}_2(\mathbf{N}(w)).$$

We consider there cases.

Case 1: $q_1 q_2 > 1$. Since $p_{\mu, \gamma}^* > q_1$, it follows that $p_{\mu, \gamma}^* q_2 > 1$. Therefore there exist $b_* > 0$ and $\varrho_* > 0$ such that for $b_1, b_2 \in (0, b_*)$ and $\varrho \in (0, \varrho_*)$ one can find $\lambda_* > 0$ and $\tilde{\lambda}_* > 0$ satisfying

$$(5.34) \quad \hat{\xi}_1(\lambda_*) = \tilde{\lambda}_* \quad \text{and} \quad \hat{\xi}_2(\tilde{\lambda}_*) = \lambda_*.$$

Case 2: $p_{\mu, \gamma}^* q_2 = 1$. In this case, there exist $a_* > 0$ such that if $a_2 \in (0, a_*)$ then for every $\varrho > 0$ and $\tilde{\varrho} > 0$ one can find $\lambda_* > 0$ and $\tilde{\lambda}_* > 0$ satisfying (5.34).

Case 3: $p_{\mu, \gamma}^* q_2 < 1$. In this case for every $\varrho > 0$ and $\tilde{\varrho} > 0$ one can find $\lambda_* > 0$ and $\tilde{\lambda}_* > 0$ such that (5.34) holds.

Hence, in any case,

$$(5.35) \quad \begin{aligned} \mathbf{M}(w) \leq \lambda_* &\implies \mathbf{N}(\mathbb{S}(w)) \leq \hat{\xi}_1(\lambda_*) = \tilde{\lambda}_* \\ \mathbf{N}(w) \leq \tilde{\lambda}_* &\implies \mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \hat{\xi}_2(\tilde{\lambda}_*) = \lambda_*. \end{aligned}$$

The rest of the proof can be proceeded as in the proof of Lemma 5.1 and the proof of Theorem F.II. and we omit it. \square

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CENTRO DE MODELAMIENTO MATEMÁTICO (UMI 2807 CNRS), UNIVERSIDAD DE CHILE, CASILLA 170 CORREO 3, SANTIAGO, CHILE.

DEPARTAMENTO DE MATEMÁTICA, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, SANTIAGO, CHILE